Lecture 23 (Sec 10.1 - Part 1)

✩ Sturm - Liouville Problems & Eigenfunction Expansions

Sep of Variables $\longrightarrow$

$$X'' + \lambda X = 0 \qquad 0 < x < L$$

with endpoint conditions

$$X(0) = X(L) = 0$$

or $\quad X'(0) = X'(L) = 0$

or $\quad X(0) = X'(L) = 0$

The solns are linear combinations of eigenfunctions X_n for each eigenvalue λ_n.

Ex: $\quad X'' + \lambda X = 0$

$X(0) = X(L) = 0$

eigenvalues $\lambda_n = \dfrac{n^2\pi^2}{L^2}$

eigenfunctions: $X_n = \sin\left(\dfrac{n\pi x}{L}\right)$

Solution: $X = \displaystyle\sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{L}\right)$

Special cases of the more general Sturm-Liouville Problem:

$$\begin{cases} \dfrac{d}{dx}\left[p(x)\dfrac{dy}{dx}\right] - q(x)y + \lambda r(x)y = 0 & (a < x < b) \\ \alpha_1 y(a) - \alpha_2 y'(a) = 0 & \alpha_1, \alpha_2 \text{ cannot both be zero} \\ \beta_1 y(b) + \beta_2 y'(b) = 0 & \beta_1, \beta_2 \text{ cannot both be zero} \end{cases}$$

Here λ - eigenvalue

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Want to find both λ and $y(x)$

Ex: $y'' + \lambda y = 0$ $\quad 0 < x < 1$

$y(0) = 0$

$y(1) = 0$

Here $p(x) = 1$, $q(x) = 0$, $r(x) = 1$

$\alpha_1 = 1 \quad \alpha_2 = 0$

$\beta_1 = 1 \quad \beta_2 = 0$

Ex: $y'' + \lambda y = 0$ $\quad 0 < x < 1$

$y'(0) = y'(1) = 0$

Here $p(x) = 1$, $q(x) = 0$, $r(x) = 1$

$\alpha_1 = 0 \quad \alpha_2 = 1$

$\beta_1 = 0 \quad \beta_2 = 1$

Note: Sturm-Liouville problem always has the trivial solution $y(x) = 0$

Goal: Find all eigenvalues that result in nontrivial solutions ($y(x) \not\equiv 0$)

Ex: Let's go back $\quad 0 < x < 1$

$$\begin{cases} y'' + \lambda y = 0 \\ y(0) = y(1) = 0 \end{cases}$$

To investigate this properly: 3 possible cases

(i) $\lambda < 0$ $\quad$ (write as $\lambda = -\alpha^2$, $\alpha > 0$)

(ii) $\lambda = 0$

(iii) $\lambda > 0$ (write as $\lambda = +\alpha^2,\ \alpha > 0$)

case (i) $\lambda = -\alpha^2 < 0$

$$\begin{cases} y'' - \alpha^2 y = 0 \\ y(0) = y(1) = 0 \end{cases}$$

$$y(x) = C_1 e^{\alpha x} + C_2 e^{-\alpha x} = C_3 \cosh(\alpha x) + C_4 \sinh(\alpha x)$$

BC: $y(0) = 0 = C_1 e^0 + C_2 e^0 \longrightarrow C_1 + C_2 = 0 \quad C_2 = -C_1$

$y(1) = 0 = C_1 e^{\alpha} + C_2 e^{-\alpha}$

$0 = C_1 (e^{\alpha} - e^{-\alpha})$

either $C_1 = 0$

$\rightarrow C_2 = 0$

$y = 0$ trivial

~~$e^{\alpha} - e^{-\alpha} = 0$~~

~~$e^{\alpha} = e^{-\alpha}$~~

~~only true if $\alpha = 0$~~

$\Rightarrow\!\Leftarrow$

assumed $\alpha > 0$

So there NO nontrivial solutions with $\lambda < 0$.

$\boxed{\text{case (ii)}}$ $\lambda = 0$

$$\begin{cases} y'' + \cancel{\lambda y}^{\,0} = y'' = 0 \\ y(0) = y(1) = 0 \end{cases}$$

$\int y'' = \int 0$

$\int y' = \int A$

$y = Ax + B$

$y(0) = 0 = A\cancel{0} + B \longrightarrow B = 0$

$y(1) = 0 = A \cdot 1 \longrightarrow A = 0$

$y = 0$

There are NO nontrivial solution when $\lambda = 0$

Case (iii) $\quad \lambda = +\alpha^2 > 0$

$$\begin{cases} y'' + \alpha^2 y = 0 \\ y(0) = y(1) = 0 \end{cases}$$

$$y = A\cos(\alpha x) + B\sin(\alpha x)$$

$$y(0) = 0 = A\cancel{\cos(0)}^{1} + B\cancel{\sin(0)}^{0} \rightarrow A = 0$$

$$y(1) = 0 = B\sin(\alpha)$$

$$\sin(\alpha) = 0$$

$$\alpha = n\pi \qquad n = 1, 2, 3, \ldots$$

$$\boxed{\begin{aligned} \lambda_n &= \alpha^2 = n^2\pi^2 \\ y_n &= \sin(n\pi x) \end{aligned}}$$ nontrivial solutions.

For this problem, the only nontrivial sol'ns occur when $\lambda > 0$

Thm (Sturm-Liouville Eigenvalues)

Suppose that $p(x)$, $p'(x)$, $q(x)$, $r(x)$ are all continuous on $[a,b]$ and that $p(x) > 0$, $r(x) > 0$ at each point in $[a,b]$

Then the eigenvalues form an increasing sequence of real numbers

$$\lambda_1 < \lambda_2 < \lambda_3 < \cdots < \lambda_n < \cdots$$

with $\lim_{n \to \infty} \lambda_n = +\infty$

Furthermore, if $q(x) \geq 0$ and $\alpha_1, \alpha_2, \beta_1, \beta_2 \geq 0$

Furthermore, if $q(x) \geq 0$ and $\alpha_1, \alpha_2, \beta_1, \beta_2 \geq 0$ then all λ's are also nonnegative ($\lambda_n \geq 0$)

<u>Ex</u>: $\begin{cases} y'' + \lambda y = 0 & 0 < x < L \\ y(0) = 0 & \\ h\,y(L) + y'(L) = 0 & h > 0 \end{cases}$

Here $p(x) = 1$, $q(x) = 0$, $r(x) = 1$

$\alpha_1 = 1 \qquad \alpha_2 = 0$

$\beta_1 = h \qquad \beta_2 = 1$

By the Thm, since $q(x), \alpha_1, \alpha_2, \beta_1, \beta_2 \geq 0$ the eigenvalues $\lambda \geq 0$ all nonnegative

1. Find the λs

$\lambda = 0$ case: $\begin{cases} y'' = 0 \\ y(0) = 0 \\ h\,y(L) + y'(L) = 0 \end{cases}$

$y = Ax + B$

$y = Ax$

$y' = A$

BCs: $y(0) = 0 = A \cdot 0 + B \longrightarrow B = 0$

$h\,y(L) + y'(L) = 0$

$h(AL) + A = 0$

$A(hL + 1) = 0 \qquad h, L > 0$

$A = 0 \qquad y(x) \equiv 0$

So there are <u>NO</u> nontrivial solns when $\lambda = 0$

$\lambda > 0$ case, let $\lambda = +\alpha^2$, $\alpha > 0$

$$\begin{cases} y'' + \alpha^2 y = 0 \\ y(0) = 0 \\ h\,y(L) + y'(L) = 0 \end{cases}$$

Soln: $y(x) = A\cos(\alpha x) + B\sin(\alpha x)$

BCs: $y(0) = 0 = A\overset{1}{\cancel{\cos(0)}} + B\overset{0}{\cancel{\sin(0)}} \to A = 0$

$$\left(y'(x) = \alpha B\cos(\alpha x) \right)$$

BC: $h\,y(L) + y'(L) = 0$

$$h\left(B\sin(\alpha L)\right) + \left(\alpha B\cos(\alpha L)\right) = 0$$

$$hB\sin(\alpha L) = -\alpha B\cos(\alpha L)$$

$$\tan(\alpha L) = \frac{\sin(\alpha L)}{\cos(\alpha L)} = -\frac{\alpha \cancel{B}}{h\cancel{B}} = -\frac{\alpha}{h}\frac{L}{L} = -\frac{(\alpha L)}{hL}$$

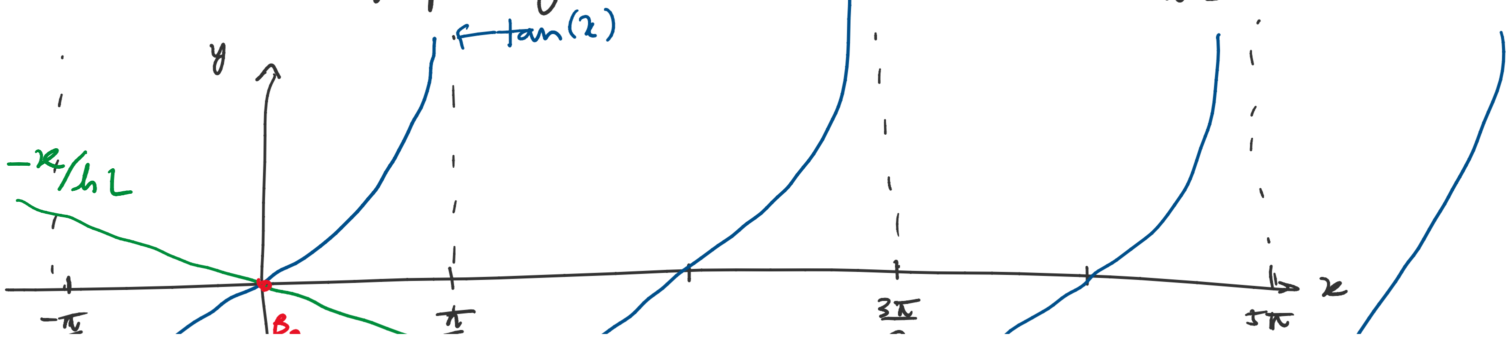
$$\tan(\alpha L) = -\frac{(\alpha L)}{hL}$$

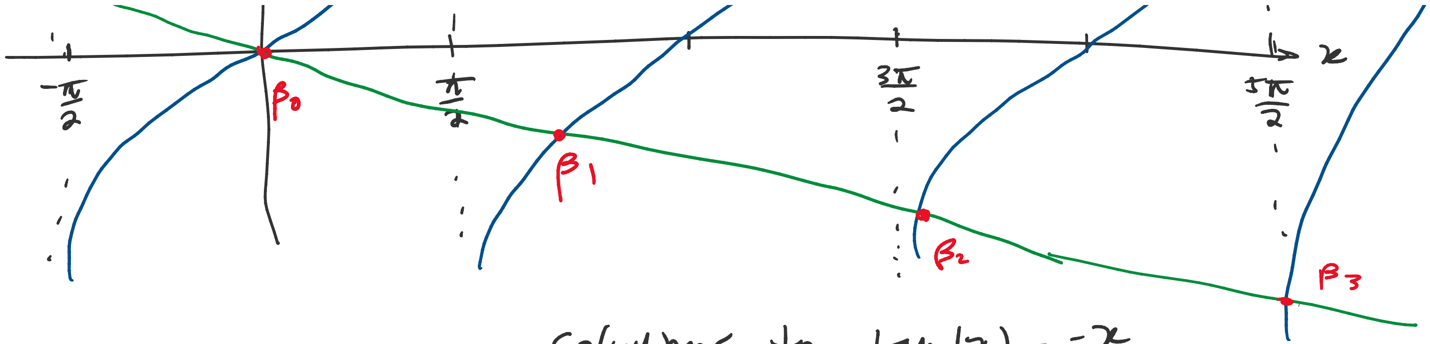
Want to find α to get $\lambda = \alpha^2$

Simplify: call $x = \alpha L$

Solve for $\underline{x}$ $\quad \tan(x) = \frac{-x}{hL}$

do this graphically by plotting (find the points x where $\tan(x)$ intersects with $-\frac{x}{hL}$





so αL are solutions to $\tan(x) = \frac{-x}{hL}$

let $\beta = \alpha L$ $\quad \beta_0, \beta_1, \beta_2, \beta_3, \dots$ are the intersections of $\tan(x)$ and $\frac{-x}{hL}$

so $\lambda_n = \alpha^2 = \left(\frac{\beta_n}{L}\right)^2$ eigenvalues

see $\lim_{n\to\infty} \beta_n \longrightarrow \left(\lim_{n\to\infty} \frac{(2n-1)\pi}{2}\right) = +\infty$

eigenfunctions $y_n = \sin(\alpha_n x) = \sin\left(\frac{\beta_n x}{L}\right)$