✩ Sturm–Liouville Problems + Eigenfunction Expansions

Sturm–Liouville Problem:

$$\begin{cases} \frac{d}{dx}\left[p(x)\frac{dy}{dx}\right] - q(x)\,y + \lambda r(x)\,y = 0 & (a<x<b) \\ \alpha_1 y(a) - \alpha_2 y'(a) = 0 \\ \beta_1 y(b) - \beta_2 y'(b) = 0 \end{cases}$$

We established that if $q(x), \alpha_1, \alpha_2, \beta_1, \beta_2 \geq 0$
then it has eigenvalues

$$0 \leq \lambda_1 < \lambda_2 < \lambda_3 < \dots < \lambda_n < \dots$$

with $\lim\limits_{n\to\infty} \lambda_n \to \infty$

where each λ_n has a eigenfunction $y_n(x)$

<u>Thm</u> (Orthogonality of Eigenfunctions)
Assume we have the same conditions as Eigenvalue Thm
Let $y_i(x)$ and $y_j(x)$ be eigenfunctions associated
with eigenvalues λ_i and λ_j. $(\lambda_i \neq \lambda_j)$

Then $$\int_a^b y_i(x)\, y_j(x) \underbrace{r(x)}_{\text{weight function}}\, dx = 0$$

<u>Ex</u>: $$\begin{cases} y'' + \lambda y = 0 & 0<x<\pi \\ y(0) = y(\pi) = 0 \end{cases}$$

Here: $p = 1,\ q = 0,\ r = 1$
$\alpha_1 = \beta_1 = 1 \qquad \alpha_2 = \beta_2 = 0$

eigenvalues: $\lambda_n = n^2$
eigenfunctions: $y_n = \sin(nx)$

Orthogonality:

Orthogonality:

Assume $\lambda_n \neq \lambda_m$

$y_n = \sin(nx) \qquad y_m = \sin(mx) \qquad r(x) = 1$

$$\int_0^{\pi} \sin(nx)\sin(mx)\,(1)\,dx = \int_0^{\pi} \frac{1}{2}\left[\cos(nx-mx) - \cos(nx+mx)\right]dx$$

$$= \frac{1}{2}\left[\frac{\sin(nx-mx)}{n-m} - \frac{\sin(nx+mx)}{n+m}\right]_0^{\pi}$$

product-to-sum trig identity

$2\sin\theta\sin\varphi = \cos(\theta-\varphi) - \cos(\theta+\varphi)$

Let $\theta = nx \quad \varphi = mx$

$$= \frac{1}{2}\left[\frac{\cancelto{0}{\sin((n-m)\pi)} - \cancelto{0}{\sin(0)}}{n-m} - \frac{\cancelto{0}{\sin((n+m)\pi)} - \cancelto{0}{\sin(0)}}{n+m}\right]$$

$$= 0$$

Conversely, if $n = m$, then

$$\int_0^{\pi} \sin^2(nx)\,dx = \int_0^{\pi} \frac{1}{2}\left[\cos(\cancelto{0}{nx-nx}) - \cos(nx+nx)\right]dx$$

$$= \frac{1}{2}\int_0^{\pi} \left(1 - \cos(2nx)\right)dx$$

$$= \frac{1}{2}\left[x - \frac{\sin(2nx)}{2n}\right]_0^{\pi}$$

$$= \frac{1}{2}\left[\pi - 0 - \frac{(\cancelto{0}{\sin(2n\pi)} - \cancelto{0}{\sin(0)})}{2n}\right] = \frac{\pi}{2}$$

proof of Thm on page 640 in Textbook.

II Eigenfunction Expansions.

If $y_n(x)$ are eigenfunctions, then we can write any function $f(x)$ as a linear combination of the y_n

$$f(x) = \sum_{n=1}^{\infty} c_n y_n(x)$$

where c_n are constant coefficients

Ex: Use the eigenfunction $y_n = \sin(nx)$

then $f(x) = \displaystyle\sum_{n=1}^{\infty} c_n \sin(nx) \longleftarrow$ Fourier Sine series

Fourier coefficients

then $f(x) = \sum_{n=1} c_n \sin(nx)$ series

So the c_n are Fourier coefficients

$$c_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin(nx)\, dx$$

<u>Note</u>: If the eigenfunctions are
$y_n = \sin\left(\frac{n\pi x}{L}\right)$ or $\cos\left(\frac{n\pi x}{L}\right) \rightarrow$ Fourier series

we already know how to compute the coefficients $c_n \longrightarrow$ Fourier coeff.

<u>Q</u>: How to compute c_n ~~if~~ the $y_n(x)$ are not Fourier series functions?

Let $y_n(x)$ be eigenfunctions (so we can assume they are orthogonal)

Expand function $f(x)$

$$f(x) = \sum_{n=1}^{\infty} c_n y_n(x)$$

Now multiply both sides by $y_m(x)\, r(x)$

$$f(x)\, y_m(x)\, r(x) = \sum_{n=1}^{\infty} c_n y_n(x)\, y_m(x)\, r(x)$$

Integrate both sides from a to b

$$\int_a^b f(x)\, y_m(x)\, r(x)\, dx = \sum_{n=1}^{\infty} c_n \underbrace{\int_a^b y_m(x)\, y_n(x)\, r(x)\, dx}_{\text{By orthogonality } \begin{cases} 0 & \text{if } n \neq m \\ \text{non zero} & \text{if } n = m \end{cases}}$$

$$= c_m \int_a^b [y_m(x)]^2\, r(x)\, dx$$

So
$$\boxed{c_m = \frac{\int_a^b f(x)\, y_m(x)\, r(x)\, dx}{\int_a^b [y_m(x)]^2\, r(x)\, dx}}$$

Check this for $y_n = \sin\left(\frac{n\pi x}{L}\right)$ on $0 < x < L$

$$\int_a^b [y_n(x)]^2 r(x)\,dx = \int_0^L \sin^2\left(\frac{n\pi x}{L}\right)(1)\,dx$$
$$= \frac{1}{2}\int_0^L \left(1 - \cos\left(\frac{2n\pi x}{L}\right)\right) dx$$
$$= \frac{1}{2}\left[x - \sin\left(\frac{2n\pi x}{L}\right)\cdot\frac{L}{2n\pi}\right]_0^L$$
$$= \frac{1}{2}\left[(L-0) - \frac{L}{2n\pi}\left[\cancel{\sin(2n\pi)}^{\,0} - \cancel{\sin(0)}^{\,0}\right]\right] = \frac{L}{2}$$

$$\int_a^b f(x)\, y_n(x)\, r(x)\,dx = \int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)(1)\,dx$$

so $$c_n = \frac{\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx}{\frac{L}{2}} = \boxed{\frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx}$$

Fourier Coefficents ✓

<u>Ex</u>: (same S-L from Lec 23)

$$\begin{cases} y'' + \lambda y = 0 & (0 < x < L) \\ y(0) = 0 & \\ h\,y(L) + y'(L) = 0 & (h > 0) \end{cases}$$

eigenvalues: $\lambda_n = \frac{\beta_n^2}{L^2}$

eigenfunctions: $y_n = \sin\left(\frac{\beta_n x}{L}\right)$

where β_n are positive solutions to $\tan(x) = -\frac{x}{hL}$

$f(x) = 1$ using these eigenfcns.

Let's expand $f(x)=1$ using these eigenfcns.

$$1 = \sum_{n=1}^{\infty} c_n \sin\left(\frac{\beta_n x}{L}\right)$$

$$c_n = \frac{\int_0^L (1)\sin\left(\frac{\beta_n x}{L}\right)(1)\,dx}{\int_0^L \sin^2\left(\frac{\beta_n x}{L}\right)(1)\,dx} = \frac{\text{top}}{\text{bottom}}$$

$$\text{top} = \int_0^L \sin\left(\frac{\beta_n x}{L}\right)dx = \left[-\cos\left(\frac{\beta_n x}{L}\right)\frac{L}{\beta_n}\right]_0^L$$

$$= -\frac{L}{\beta_n}\left[\cos(\beta_n) - \cancelto{1}{\cos(0)}\right] = \frac{L}{\beta_n}\left[1-\cos(\beta_n)\right]$$

$$\text{bottom} = \int_0^L \sin^2\left(\frac{\beta_n x}{L}\right)dx = \int_0^L \frac{1}{2}\left[1-\cos\left(\frac{2\beta_n x}{L}\right)\right]dx$$

$$= \frac{1}{2}\left[x - \sin\left(\frac{2\beta_n x}{L}\right)\frac{L}{2\beta_n}\right]_0^L$$

$$= \frac{1}{2}\left[(L-0) - \left(\sin(2\beta_n) - \cancelto{0}{\sin(0)}\right)\frac{L}{2\beta_n}\right]$$

$$= \frac{1}{2}\left[L - \frac{L}{2\beta_n}\sin(2\beta_n)\right] \qquad \sin(2\theta) = 2\sin(\theta)\cos(\theta)$$

$$= \frac{1}{2}\left[\frac{L\beta_n}{\beta_n} - \frac{L}{\cancel{2}\beta_n}\cdot\cancel{2}\sin(\beta_n)\cos(\beta_n)\right]$$

$$= \frac{L}{2\beta_n}\left[\beta_n - \sin(\beta_n)\cos(\beta_n)\right]$$

$$c_n = \frac{\frac{\cancel{L}}{\cancel{\beta_n}}\left[1-\cos(\beta_n)\right]}{\frac{\cancel{L}}{2\cancel{\beta_n}}\left[\beta_n - \sin(\beta_n)\cos(\beta_n)\right]} = \boxed{\frac{2\left[1-\cos(\beta_n)\right]}{\beta_n - \sin(\beta_n)\cos(\beta_n)}}$$

$$\boxed{1 = \sum_{n=1}^{\infty} \left(\frac{2\left[1-\cos(\beta_n)\right]}{\beta_n - \sin(\beta_n)\cos(\beta_n)} \right) \sin\left(\frac{\beta_n x}{L}\right)}$$

Eigenfunction Expansion.