Lecture 2 (Sec 5.2)

Part 1

☆ The Eigenvalue Method for Homogeneous Systems

A homogeneous system of linear differential eqns:

$$\underline{x}' = \underline{\underline{A}}\,\underline{x}$$

Here $\underline{\underline{A}}$ is a constant matrix

Recall: For the 1st order linear eqn

$$x'(t) = \lambda x \quad \rightarrow \quad x(t) = x_0 e^{\lambda t}$$

For the 2nd order linear eqn:

$$a x'' + b x' + c x = 0$$

We assume solns look like $x = e^{rt}$
which leads to the characteristic eqn:
$$a r^2 + b r + c = 0 \quad \rightarrow \text{roots } r_1, r_2$$

and general soln $x(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$

We will try something similar for systems:

Assume solutions of the form:

$$\underline{x} = e^{\lambda t}\,\underline{v} = e^{\lambda t}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

Plug into our system:

$$\underline{x}' = \lambda e^{\lambda t}\underline{v} \doteq \underline{\underline{A}}\,\underline{x} = \underline{\underline{A}}\left(e^{\lambda t}\underline{v}\right)$$

$$\rightarrow \qquad \lambda\underline{v} = \underline{\underline{A}}\,\underline{v}$$

Rewrite as: $\underbrace{(\underline{\underline{A}} - \lambda\underline{\underline{I}})}_{\text{solve for } \lambda}\underline{v} = \underline{0}$ where $\underline{\underline{I}} = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$ identity matrix

This is the vector equivalent of the characteristic eqn.

λ is called an <u>eigenvalue</u>

$\underline{v}$ is called an <u>eigenvector</u>

<u>Ex</u>: $\underline{x}' = \begin{pmatrix} 0 & 1 \\ 3 & 2 \end{pmatrix}\underline{x}$ $\qquad (*)$

Solns look like: $\underline{x} = e^{\lambda t}\underline{v}$

Need to solve: $(\underline{\underline{A}} - \lambda\underline{\underline{I}})\underline{v} = 0$

This system has a solution when

$$\det(\underline{\underline{A}} - \lambda\underline{\underline{I}}) = 0$$

$$\det\begin{pmatrix} 0-\lambda & 1 \\ 3 & 2-\lambda \end{pmatrix} = -\lambda(2-\lambda) - 1\cdot 3$$

$$= \boxed{\lambda^2 - 2\lambda - 3 = 0}$$

characteristic eqn

$$\lambda^2 - 2\lambda - 3 = 0$$

$$(\lambda - 3)(\lambda + 1) = 0 \longrightarrow \lambda = 3, -1$$

For each eigenvalue there is a corresponding eigenvector

$$\lambda_1 = 3 \longleftrightarrow \underline{v}^{(1)} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

To find $\underline{v}^{(1)}$ plug back into char eqn.

$$(\underline{\underline{A}} - \lambda_1 \underline{\underline{I}})\, \underline{v}^{(1)} = \underline{0}$$

$$\begin{bmatrix} 0-\lambda_1 & 1 \\ 3 & 2-\lambda_1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -3 & 1 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$-3v_1 + v_2 = 0 \longrightarrow v_2 = 3v_1$$

$$3v_1 - v_2 = 0 \longrightarrow v_2 = 3v_1$$

So here, v_1 is a free variable

$$\underline{v}^{(1)} = \begin{bmatrix} v_1 \\ 3v_1 \end{bmatrix}$$

Choose any value for v_1

for simplicity, take $v_1 = 1$

$$\underline{v}^{(1)} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \quad \lambda_1 = 3$$

Show that $\underline{x}^{(1)} = e^{\lambda_1 t}\,\underline{v}^{(1)} = e^{3t}\begin{bmatrix}1\\3\end{bmatrix}$

Solves the system (*)

$$\underline{x}' = \begin{bmatrix}0 & 1\\3 & 2\end{bmatrix}\underline{x}$$

$$3e^{3t}\begin{bmatrix}1\\3\end{bmatrix} \stackrel{?}{=} \begin{bmatrix}0 & 1\\3 & 2\end{bmatrix}\left(e^{3t}\begin{bmatrix}1\\3\end{bmatrix}\right)$$

$$e^{3t}\begin{bmatrix}3\\9\end{bmatrix} \stackrel{?}{=} e^{3t}\begin{bmatrix}0+1\cdot 3\\3\cdot 1+2\cdot 3\end{bmatrix} = e^{3t}\begin{bmatrix}3\\9\end{bmatrix}$$

so yes, $\underline{x}^{(1)}$ is a fundamental solution of (*)

The second fundamental soln is:

$\underline{x}^{(2)} = e^{\lambda_2 t}\,\underline{v}^{(2)}$ where $\lambda_2 = -1$, $\underline{v}^{(2)} = \begin{bmatrix}v_1\\v_2\end{bmatrix}$

<u>Exercise</u>: find the eigenvector $\underline{v}^{(2)}$ corresponding to $\lambda_2 = -1$

Then the general solution is:

$$\underline{x}(t) = c_1 e^{\lambda_1 t}\,\underline{v}^{(1)} + c_2 e^{\lambda_2 t}\,\underline{v}^{(2)}$$

$$= c_1 e^{3t}\begin{bmatrix}1\\3\end{bmatrix} + c_2 e^{-t}\begin{bmatrix}v_1\\v_2\end{bmatrix}$$

Summary:

Given $\underline{x}' = \underline{\underline{A}}\,\underline{x}$ $\quad (\square)$

1. Plug $\underline{x} = e^{\lambda t}\underline{v}$ into $(\square)$ to obtain the eigenvalue problem: $\underline{\underline{A}}\,\underline{v} = \lambda \underline{v}$

2. Find n eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ by solving $\det(\underline{\underline{A}} - \lambda \underline{\underline{I}}) = 0$

3. Find n eigenvectors $\underline{v}^{(1)}, \underline{v}^{(2)}, \ldots, \underline{v}^{(n)}$ by solving $(\underline{\underline{A}} - \lambda_i \underline{\underline{I}})\underline{v}^{(i)} = \underline{0}$

4. The general solution is:

$$\underline{x}(t) = c_1 e^{\lambda_1 t}\underline{v}^{(1)} + \ldots + c_n e^{\lambda_n t}\underline{v}^{(n)}$$

<u>Note</u>: The first example had real, distinct eigenvalues $\lambda_1 = 3$ and $\lambda_2 = -1$

<u>Q</u>: What happens when the eigenvalues are complex?

$\boxed{\text{Part 2}}$

complex eigenvalues

Ex: $\underline{x}' = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \underline{x}$

eigenvalues: $\det(\underline{\underline{A}} - \lambda \underline{\underline{I}}) = 0$

$$\begin{vmatrix} 1-\lambda & 1 \\ -1 & 1-\lambda \end{vmatrix} = (1-\lambda)^2 + 1 = 0$$

$$(1-\lambda)^2 = -1$$

Complex eigenvalues always show up in conjugate pairs $\longrightarrow$

$$1-\lambda = \pm i$$
$$\lambda = 1 \pm i$$

eigenvectors: $(\underline{\underline{A}} - \lambda \underline{\underline{I}}) \underline{v} = \underline{0}$

$\lambda_1 = 1+i$

$$\begin{bmatrix} -i & 1 \\ -1 & -i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$-iv_1 + v_2 = 0 \longrightarrow v_2 = iv_1$$
$$v_1 - iv_2 = 0 \longrightarrow v_2 = \frac{-v_1}{i} = iv_1$$

so v_1 is a free variable, choose $v_1 = 1$

$$\underline{v}^{(1)} = \begin{bmatrix} v_1 \\ iv_1 \end{bmatrix} = \begin{bmatrix} 1 \\ i \end{bmatrix}$$

$\lambda_2 = 1 - i$

$$\begin{bmatrix} i & 1 \\ -1 & i \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$iv_1 + v_2 = 0 \longrightarrow v_2 = -iv_1$$

$$-v_1 + iv_2 = 0 \longrightarrow v_2 = \frac{v_1}{i} = -iv_1$$

so v_1 is a free variable, choose $v_1 = 1$

$$\underline{v}^{(2)} = \begin{bmatrix} v_1 \\ -iv_1 \end{bmatrix} = \begin{bmatrix} 1 \\ -i \end{bmatrix}$$

So we have the general solution:

$$\underline{x}(t) = c_1 e^{(1+i)t}\begin{bmatrix} 1 \\ i \end{bmatrix} + c_2 e^{(1-i)t}\begin{bmatrix} 1 \\ -i \end{bmatrix}$$

<u>Note</u>: eigenvectors are also conjugate pairs

Rewrite:

$$e^{(1+i)t}\begin{bmatrix} 1 \\ i \end{bmatrix} = e^{t}e^{it}\begin{bmatrix} 1 \\ i \end{bmatrix} \qquad e^{it} = \cos t + i\sin t$$

$$= e^{t}(\cos t + i\sin t)\begin{bmatrix} 1 \\ i \end{bmatrix}$$

$$= e^{t}\begin{bmatrix} \cos t + i\sin t \\ i\cos t - \sin t \end{bmatrix}$$

$$= e^{t}\left\{ \underbrace{\begin{bmatrix} \cos t \\ -\sin t \end{bmatrix}}_{\underline{u}} + i\underbrace{\begin{bmatrix} \sin t \\ \cos t \end{bmatrix}}_{\underline{v}} \right\}$$

$$e^{(1-i)t}\begin{bmatrix}1\\-i\end{bmatrix} = e^{t}\left\{\underline{u} - i\underline{v}\right\}$$

We want real-valued solutions, so: general solution

$$\underline{x}(t) = c_1 e^{Re(\lambda)}\underline{u} + c_2 e^{Re(\lambda)}\underline{v}$$

$$\boxed{\underline{x}(t) = c_1 e^{t}\begin{bmatrix}\cos t\\-\sin t\end{bmatrix} + c_2 e^{t}\begin{bmatrix}\sin t\\\cos t\end{bmatrix}}$$

☆ Phase Portraits:

For a 2D system, we can represent the solution graphically in the phase plane.

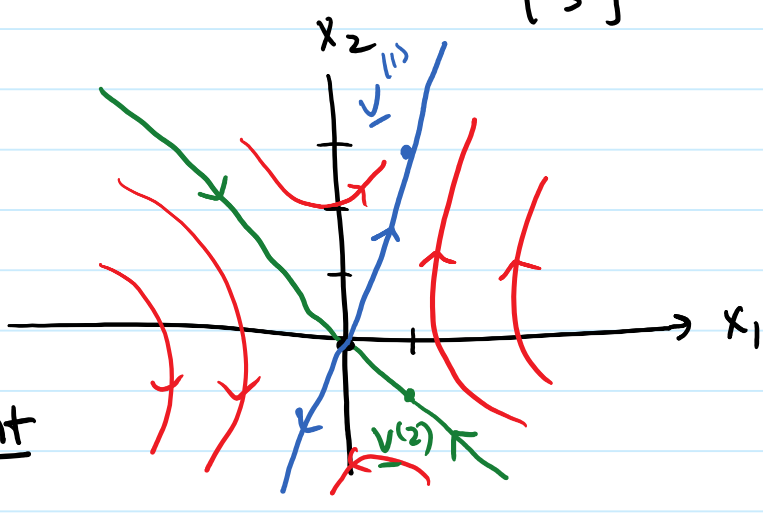
Ex: $\underline{x}' = \begin{bmatrix}0 & 1\\3 & 2\end{bmatrix}\underline{x}$

has eigenvalues: $\lambda_1 = 3$ $\quad\quad$ $\lambda_2 = -1$

and corresponding eigenvectors: $\underline{v}^{(1)} = \begin{bmatrix}1\\3\end{bmatrix}$ $\quad$ $\underline{v}^{(2)} = \begin{bmatrix}1\\-1\end{bmatrix}$

phase plane

This is called a saddle point



point out since $\lambda_1 = 3 > 0$

point in since $\lambda_2 = -1 < 0$

Draw curves that follow the arrows

Ex 2: complex eigenvalues

$$\underline{x}' = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \underline{x}$$

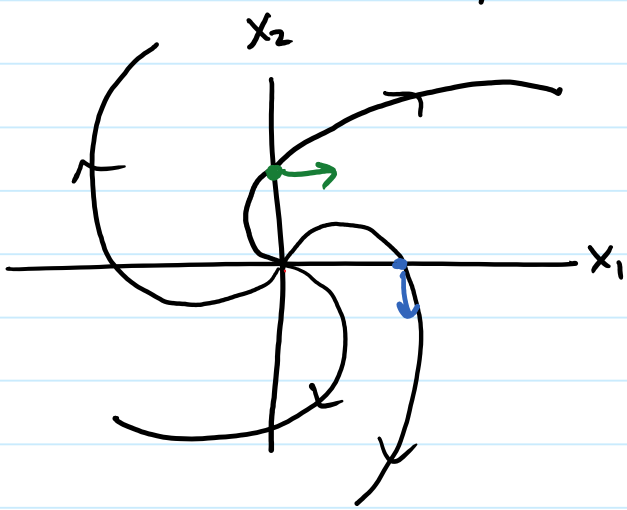
$\lambda_1 = 1+i$ $\qquad\qquad$ $\lambda_2 = 1-i$

$\underline{u} = \begin{bmatrix} \cos t \\ -\sin t \end{bmatrix}$ $\qquad\qquad$ $\underline{v} = \begin{bmatrix} \sin t \\ \cos t \end{bmatrix}$

When the eigenvalues are complex, the resulting phase portrait is a spiral

if $\quad Re(\lambda) > 0 \rightarrow$ spiral out

$\quad\quad Re(\lambda) < 0 \rightarrow$ spiral in



this is a <u>spiral</u>

$\underline{u} = \begin{bmatrix} \cos t \\ -\sin t \end{bmatrix}$ $\quad$ $\underline{u}' = \begin{bmatrix} -\sin t \\ -\cos t \end{bmatrix}$

$\underline{v} = \begin{bmatrix} \sin t \\ \cos t \end{bmatrix}$ $\quad$ $\underline{v}' = \begin{bmatrix} \cos t \\ -\sin t \end{bmatrix}$

$\underline{u}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\quad$ $\underline{u}'(0) = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$

$\underline{v}(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ $\quad$ $\underline{v}'(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$