Lecture 5 (Sec 6.1)

☆ Nonlinear Systems and Phenomena: (Chap 6)

6.1: Stability in the Phase Plane:

A system of equations

$$x' = F(x,y)$$
$$y' = G(x,y) \qquad (*)$$

is called <u>autonomous</u> if the RHS has no t dependence.

A <u>critical point</u> of system $(*)$ is a point (x_*, y_*) where both

$$F(x_*, y_*) = 0 \quad \text{and} \quad G(x_*, y_*) = 0$$

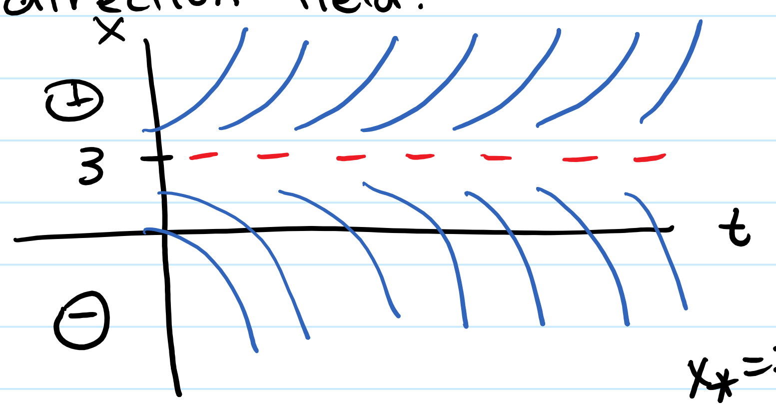
(Can also call this an equilibrium soln)

<u>Recall</u>: in 1D, we had critical points and drew direction fields to graphically represent solution curves

<u>Ex</u>: $x' = x - 3$

has a critical point @ $x_* = 3$

direction field:



when $x > 3$, x' is ⊕

when $x < 3$, x' is ⊖

$x_* = 3$ is <u>unstable</u>

Now, we want to do something similar in 2D

	1D	2D
critical points	x_*	(x_*, y_*)
graph	direction field (x vs. t)	phase portrait (y vs. x)

For both, we can evaluate the stability of the critical point by looking at the graph.

Ex: (2D)

$$x' = x - 3$$
$$y' = x + 5y + 2$$

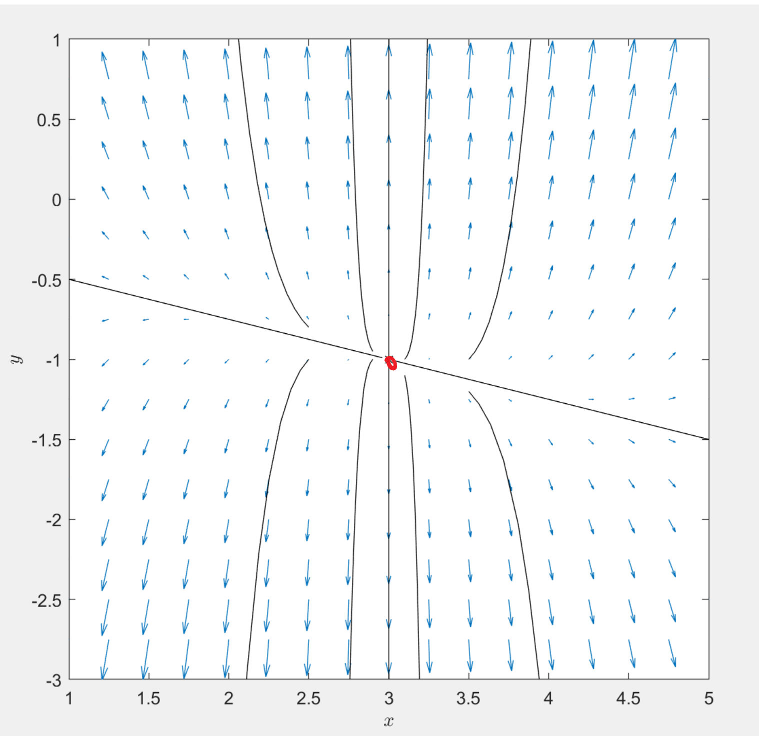
critical points:

$$x_* - 3 = 0$$
$$x_* = 3$$

$\longrightarrow$

$$x_* + 5y_* + 2 = 0$$
$$3 + 5y_* + 2 = 0$$
$$5y_* = -5$$
$$y_* = -1$$

Cp: $(3, -1)$



Use computer to draw the phase portrait

See that $(3,-1)$ is an improper nodal source

So $(3,-1)$ is an unstable critical point.

Note: we have seen all the possible behaviors of a critical point

- (im) proper nodal source/sink
- saddle point
- spiral source / sink
- center
- parallel lines.

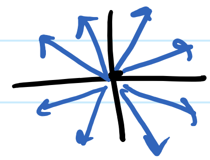
These are the same in nonlinear systems

From the phase portrait, we can determine the stability of the critical point

Three cases:

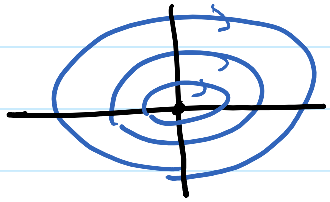
1. unstable: solutions go away (diverge) from the critical point

   Ex: proper nodal source

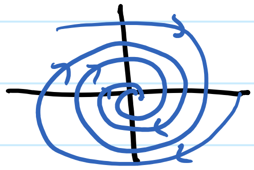
2. stable: solutions stay close to the critical point

   Ex: center

3. Asymptotically stable: as $t \to \infty$, then solutions converge to the critical point *

   Ex: spiral sink

* <u>caveat</u>: for asymptotically stable cp

Not every solution curve will converge to the critical point, but if you start "close enough" to (x_*, y_*) then you will converge.

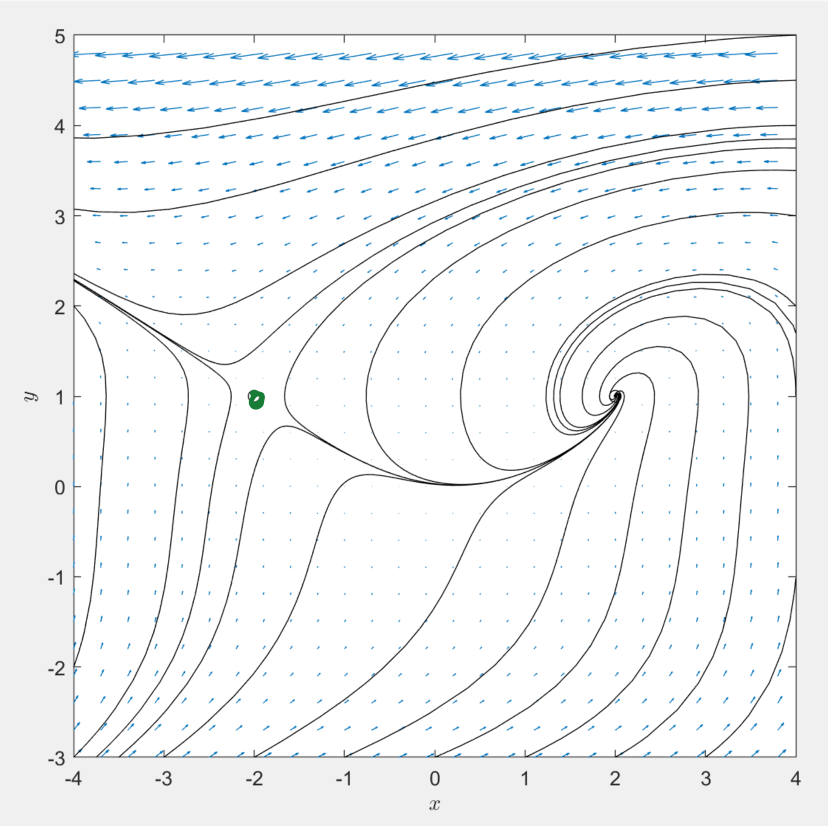
Same goes for stable c.p.

<u>Ex.</u>
$$x' = 1 - y^3$$
$$y' = x^2 - 4y$$

critical points solve $1 - y_*^3 = 0$ $\quad x_*^2 - 4y_* = 0$

So $(2,1)$ and $(-2,1)$ are cp.

<u>Note</u>: Nonlinear systems usually have multiple critical points



$(2,1)$ is a <u>spiral</u> <u>sink</u>

So it is <u>asymptotically stable</u>

$(-2,1)$ is a <u>saddle point</u>

so it is <u>unstable</u>

Not every soln curve converges to $(2,1)$, but the nearby ones do.

We can use phase planes and critical points to analyze higher order scalar ODEs

Ex: $x'' + 2x + x^2 = 0$ 2nd order scalar

Convert to a 2D system

Let $y = x'$ $x'' = y' = -2x - x^2$

So $\begin{cases} x' = y \\ y' = -2x - x^2 \end{cases}$

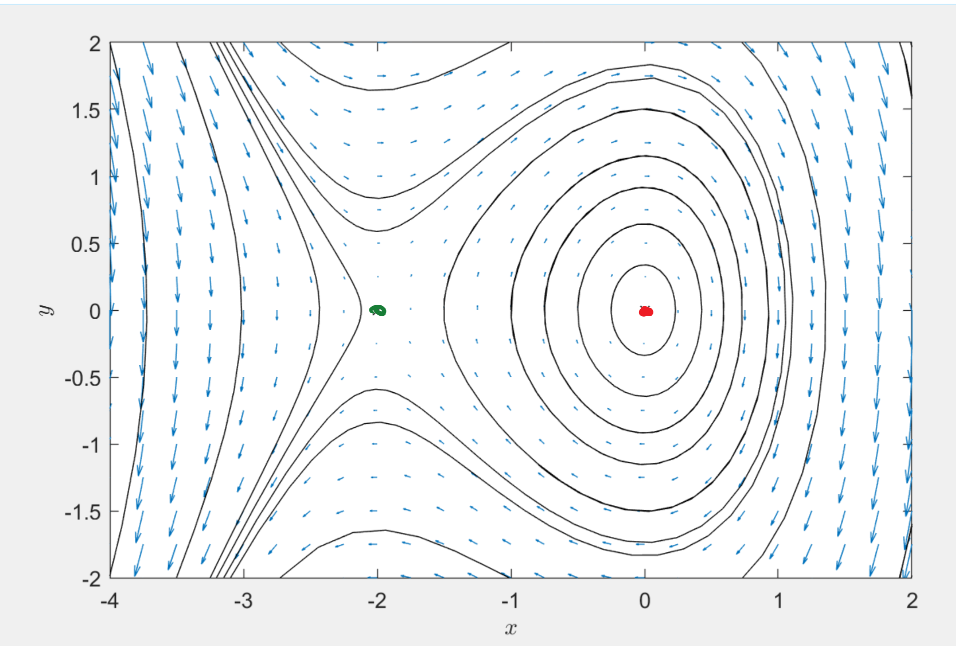
this has critical points when

$y_* = 0$ $-2x_* - x_*^2 = 0$

$-x_*(2 + x_*) = 0$

$x_* = 0, -2$

so c.p. are $(0,0)$ and $(-2,0)$



$(0,0)$ is a center so it is stable

$(-2,0)$ is a saddle point so it is unstable

We just went 1D → 2D

We can also go the other way: 2D → 1D

<u>Ex</u>: $x' = F(x,y) = 10y$
$y' = G(x,y) = -x$

The critical point is $(0,0)$

We can convert this into a 1st order ODE

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{G(x,y)}{F(x,y)} = \frac{-x}{10y}$$

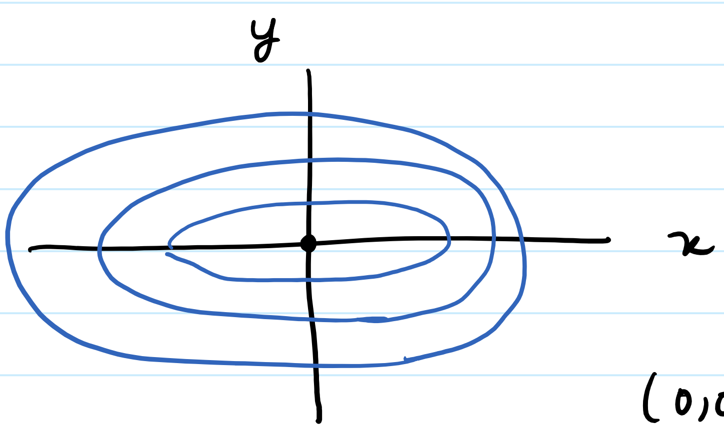
$$\frac{dy}{dx} = -\frac{x}{10y}$$ this is separable

$$\int 10y\,dy = \int -x\,dx$$

$$\frac{10y^2}{2} = -\frac{x^2}{2} + C_1$$

$$10y^2 + x^2 = C_2$$ ← this is the equation of an ellipse

So the phase portrait looks like



$$y^2 + \frac{x^2}{10} = C$$

so the ellipse is stretched in the x-direction.

$(0,0)$ is a <u>center</u> and is <u>stable</u>