☆ Linear and Almost Linear Systems

I. Stability of Linear Systems:

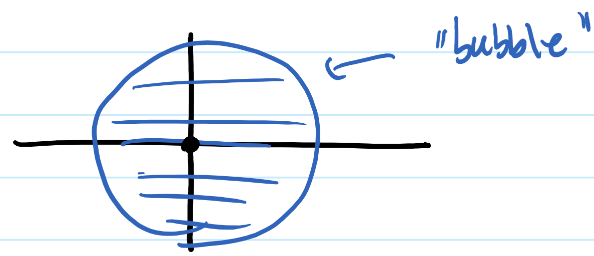
Let's go back to the 2D linear system:

$$\begin{bmatrix} x \\ y \end{bmatrix}' = \underbrace{\begin{bmatrix} a & b \\ c & d \end{bmatrix}}_{\underset{=}{A}} \begin{bmatrix} x \\ y \end{bmatrix}$$

Some observations:

- the origin $(0,0)$ is always a critical pt
- If $\det(\underset{=}{A}) = ad - bc \neq 0$, then the origin is an isolated critical point

(means there is a small "bubble" around $(0,0)$ where it is the only c.p.



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The characteristic equation is

$$\det(\underset{=}{A} - \lambda \underset{=}{I}) = \begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} = (a-\lambda)(d-\lambda) - bc$$

$$= \lambda^2 - (a+d)\lambda + ad - bc = 0$$

Let $D = \det(\underset{=}{A}) = ad - bc$

$T = \text{trace}(A) = a + d$

Then the characteristic eqn is:

$$\lambda^2 - T\lambda + D = 0$$

and the eigenvalues are:

$$\lambda = \frac{T}{2} \pm \frac{1}{2}\sqrt{T^2 - 4D}$$

Note: $\lambda = 0$ only if $D = 0$

Furthermore, we can classify the critical point $(0,0)$ as follows:

λ	Type of c.p.	Stable?
real, distinct same sign	improper node	yes, if both $\lambda < 0$ (asymp. stable)
real, distinct opposite sign	saddle point	NO
real and equal	proper or improper node	yes if $\lambda < 0$ (asymp. stable)
complex conjugates	spiral point	yes if $Re(\lambda) < 0$ (asymp. stable)
pure imaginary	center	yes

We have the following theorem to summarize:

Thm: (Stability of Linear Systems)
Let λ_1 and λ_2 be the eigenvalues of the system

$$\begin{bmatrix} x \\ y \end{bmatrix}' = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \quad \text{where } \det(\underline{A}) \neq 0.$$

Then the critical point $(0,0)$ is:

1. Asymptotically stable if $Re(\lambda) < 0$
2. Stable if $Re(\lambda) = 0$
3. unstable otherwise

So just by looking at the eigenvalues, we can assess the stability of the origin.

Ex: $$\begin{bmatrix} x \\ y \end{bmatrix}' = \begin{bmatrix} -1 & -1 \\ 6 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$D = \det(\underline{A}) = (-1)(2) - (-1)(6) = -2 + 6 = 4$$

$D \neq 0$ so $(0,0)$ is an isolated c.p.

$$T = \text{trace}(\underline{A}) = -1 + 2 = 1$$

Then
$$\lambda^2 - T\lambda + D = 0$$
$$\lambda^2 - \lambda + 4 = 0$$

$$\boxed{\lambda = \frac{1}{2} \pm \frac{\sqrt{15}}{2} i}$$

So $Re(\lambda) = \frac{1}{2} > 0 \longrightarrow$ unstable

and $(0,0)$ is a spiral source

Ex: $\begin{bmatrix} x \\ y \end{bmatrix}' = \begin{bmatrix} -4 & 1 \\ -6 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$

$D = \det(\underline{\underline{A}}) = -4(2) - 1(-6) = -8 + 6 = -2$

$D \neq 0$ so $(0,0)$ is isolated

$T = \text{trace}(\underline{\underline{A}}) = -4 + 2 = -2$

So $\lambda^2 + 2\lambda - 2 = 0$

$$\lambda = -\frac{2}{2} \pm \frac{\sqrt{(2)^2 - 4 \cdot 1 \cdot (-2)}}{2}$$

$$\lambda = -1 \pm \sqrt{3}$$

$\lambda_1 = -1 + \sqrt{3} > 0$ $\qquad$ $\lambda_2 = -1 - \sqrt{3} < 0$

so $(0,0)$ is a <u>saddle point</u>
and <u>unstable</u>

II. Small Perturbations:

What happens if we perturb a system by a little bit ε? (assume $\varepsilon \ll 1$)

Ex: $\begin{bmatrix} x \\ y \end{bmatrix}' = \begin{bmatrix} -1 & -1+\varepsilon \\ 6 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$ $\quad$ (this is our first example perturbed i.e. unstable spiral source)

$D = \det(\underline{\underline{A}}) = (-1)2 - 6(-1+\varepsilon) = -2 + 6 - 6\varepsilon$
$= 4 - 6\varepsilon$ $\qquad$ $D \neq 0$ so $(0,0)$ isolated

$T = \text{trace}(\underline{\underline{A}}) = -1 + 2 = 1$

so the characteristic eqn:

$$\lambda^2 - T\lambda + D = \lambda^2 - \lambda + 4 - 6\varepsilon = 0$$

$$\lambda = \frac{1}{2} \pm \frac{1}{2}\sqrt{(-1)^2 - 4\cdot 1\cdot(4-6\varepsilon)}$$

$$= \frac{1}{2} \pm \frac{1}{2}\sqrt{1 - 16 + 24\varepsilon}$$

$$= \frac{1}{2} \pm \frac{1}{2}\sqrt{-15 + 24\varepsilon}$$

if $\varepsilon \ll 1$, then $-15 + 24\varepsilon < 0$

So $\lambda = \frac{1}{2} \pm bi$

So $(0,0)$ remains unstable and a spiral source

(To change the type of c.p., we'd need:

$$-15 + 24\varepsilon > 0$$

$$\varepsilon > \frac{15}{24} = \frac{5}{8}$$

but $\varepsilon \ll 1$

$\Rightarrow\Leftarrow$)

Note: In most cases, a small perturbation to the Coefficients of $\underset{=}{A}$ will not change the stability of the system.

However, a perturbation can change the type of critical point
(see W3)

III. Almost Linear Systems:

Recall, the Taylor series expansion of a function of 2 variables $f(x,y)$ at the origin $(0,0)$ is:

$$f(x,y) = f(0,0) + f_x(0,0)x + f_y(0,0)y + r(x,y)$$

Here $r(x,y)$ is the remainder
$r(x,y) \sim O(x^2+y^2)$

→ if x and y are small, then $r(x,y)$ is an order of magnitude smaller

We want to use Taylor series to linearize a nonlinear system.

Ex: $x' = F(x,y)$
$y' = G(x,y)$

If $(0,0)$ is a critical point, then by definition $F(0,0) = 0 = G(0,0)$

Now, let's Taylor expand both $F(x,y)$ and $G(x,y)$ around the origin

$$x' = F(x,y) = \underbrace{\cancel{F(0,0)}}_{0} + F_x(0,0)\,x + F_y(0,0)\,y + \underbrace{\cancel{r(x,y)}}_{\text{small}}$$

$$y' = G(x,y) = \underbrace{\cancel{G(0,0)}}_{0} + G_x(0,0)\,x + G_y(0,0)\,y + \underbrace{\cancel{s(x,y)}}_{\text{small}}$$

So we can write this as:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \underbrace{\begin{bmatrix} F_x(0,0) & F_y(0,0) \\ G_x(0,0) & G_y(0,0) \end{bmatrix}}_{\underline{\underline{J}} \text{ - Jacobian matrix}} \begin{bmatrix} x \\ y \end{bmatrix} + \underbrace{\begin{bmatrix} r(x,y) \\ s(x,y) \end{bmatrix}}_{\substack{\text{very small} \\ \text{near } (0,0)}} \quad (*)$$

We call $(*)$ an <u>almost linear system</u>

The associated linear system is:

$$\begin{bmatrix} x \\ y \end{bmatrix}' = \underline{\underline{J}} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \quad (\square)$$

We can consider the almost linear system $(*)$ a perturbation of the linear system $(\square)$

<u>Q</u>: What can the linear system $(\square)$ tell us about the almost linear system $(*)$?

<u>Thm</u>: The almost linear system $(*)$ will have the same type of critical point and same stability as the linear system $(\square)$

<u>UNLESS</u> : $\lambda_1 = \lambda_2$ or $\lambda = \pm bi$

Ex: $x' = 4x + 7y - x^2 + 3y^2$
$y' = x - 2y - 13xy$

$(0,0)$ is a critical point

Jacobian:
$$\underline{\underline{J}} = \begin{bmatrix} F_x(0,0) & F_y(0,0) \\ G_x(0,0) & G_y(0,0) \end{bmatrix}$$

$$F_x\big|_{(0,0)} = 4 + 0 - 2x + 0 \big|_{(0,0)} = 4$$

$$F_y\big|_{(0,0)} = 0 + 7 + 0 + 6y \big|_{(0,0)} = 7$$

$$G_x\big|_{(0,0)} = 1 - 0 - 13y \big|_{(0,0)} = 1$$

$$G_y\big|_{(0,0)} = 0 - 2 - 13x \big|_{(0,0)} = -2$$

So $\underline{\underline{J}} = \begin{bmatrix} 4 & 7 \\ 1 & -2 \end{bmatrix}$

So the associated linear system is:
$$\begin{bmatrix} x \\ y \end{bmatrix}' = \begin{bmatrix} 4 & 7 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Note: this is just the linear terms from the original sys.

Find the eigenvalues:
$D = \det(\underline{\underline{J}}) = 4(-2) - 1(7) = -8 - 7 = -15$
$T = \text{trace}(\underline{\underline{J}}) = 4 - 2 = 2$

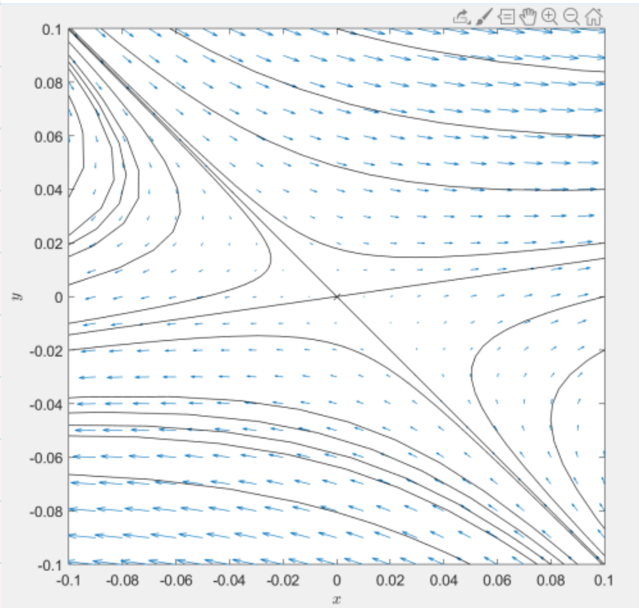
$$\lambda^2 - 2\lambda - 15 = 0$$
$$(\lambda - 5)(\lambda + 3) = 0 \qquad \lambda = 5, -3$$

$\lambda = 5, -3$

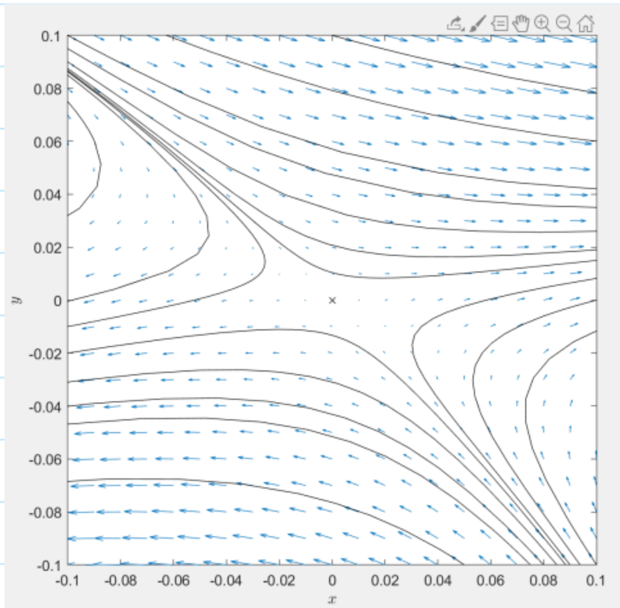
So $(0,0)$ is a saddle point and unstable for the linear system.

Likewise, $(0,0)$ is a saddle point and unstable for the almost linear system

See this in the phase portraits.

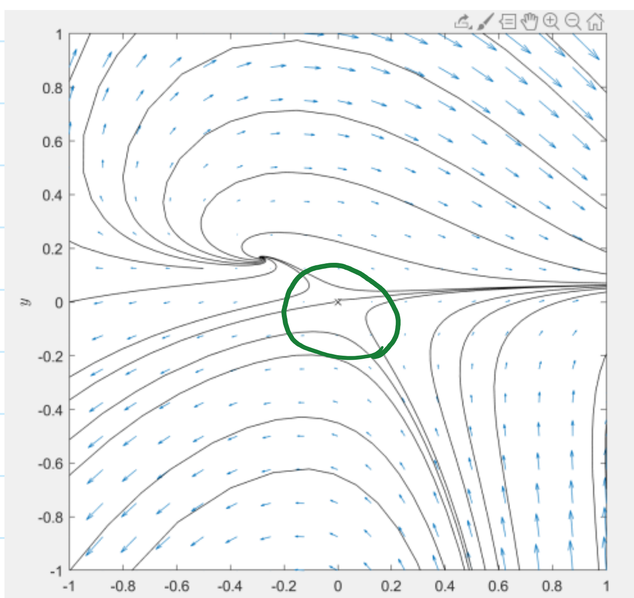


Linear System



Almost Linear System

If we zoom out on the almost linear system, we see there is another critical point as well



– bubble shows $(0,0)$ is indeed an isolated critical point

Ex: $\begin{aligned} x' &= \boxed{-6x + 4y} - x^2 + 3y^2 \\ y' &= \boxed{-4x + 2y} - 20xy \end{aligned}$ almost linear sys.

just take the linear terms

The associated linear system is:

$$\begin{bmatrix} x \\ y \end{bmatrix}' = \begin{bmatrix} -6 & 4 \\ -4 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$D = \det(\underline{A}) = -6(2) - 4(-4) = -12 + 16 = 4$$

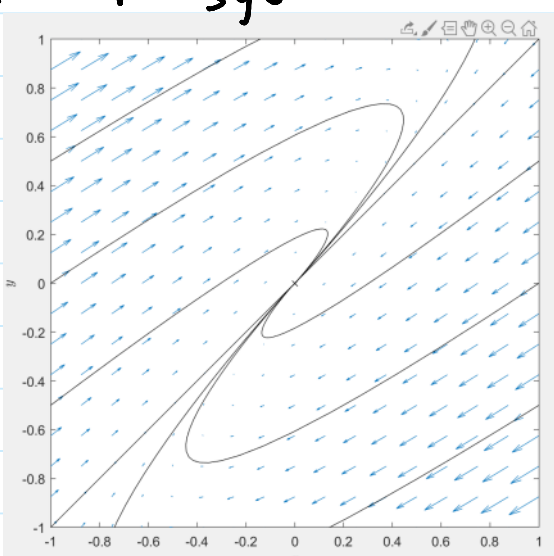
$$T = \operatorname{trace}(\underline{A}) = -6 + 2 = -4$$

$$\lambda^2 + 4\lambda + 4 = 0$$

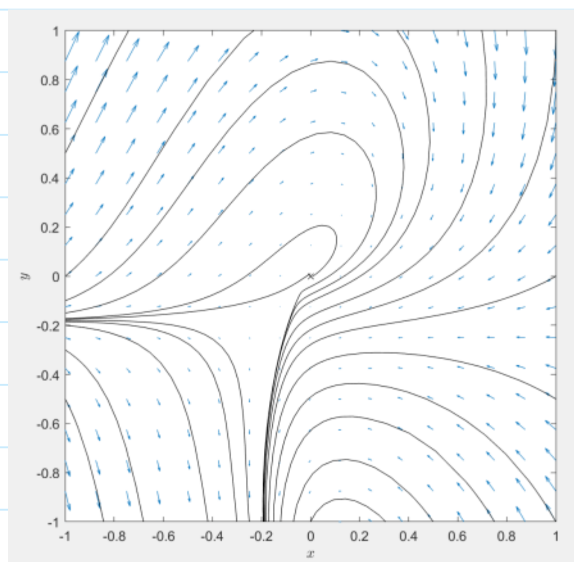
$$(\lambda + 2)^2 = 0 \quad \text{improper nodal sink}$$

$$\lambda = -2 \quad \leftarrow \text{asymptotically stable}$$

However, according to the Thm, since $\lambda_1 = \lambda_2$ we can't say anything about the almost linear system.



Linear system



Almost linear system

In fact, the almost linear system is <u>unstable</u>