Lecture 7 (Sec 6.3)

# ☆ Ecological Models: Predators & Competitors

## I. Predator - Prey Model

Consider a model of two species populations
$x(t)$ - population of prey (rabbits)
$y(t)$ - population of predators (foxes)

Alone, each population has exponential growth or decay

Assume $a, b > 0$

$$\frac{dx}{dt} = ax \qquad \frac{dy}{dt} = -by$$

rabbit pop. grows — fox pop. decays

When both rabbits and foxes are in the same habitat:

- rabbit pop declines (eaten by foxes)
- fox pop grows (increased food supply)
- both changes are proportional to each other

From these rules, we can write the Predator - Prey model

$$\begin{cases} \dfrac{dx}{dt} = ax - pxy = x(a - py) \\ \dfrac{dy}{dt} = -by + qxy = y(-b + qx) \end{cases}$$

Ex: $x' = 10x - 2xy$
$y' = -21y + 3xy$

We can use our knowledge of almost linear systems to analyze this model

1. Find the critical points:

$10x - 2xy = 0$ $\qquad$ $-21y + 3xy = 0$

$x(10 - 2y) = 0$ $\qquad$ $y(-21 + 3x) = 0$

if $x = 0 \rightarrow y = 0$

if $y = 5 \rightarrow -21 + 3x = 0$

$x = 7$

$\boxed{\begin{array}{l} (0,0) \text{ is a c.p.} \\ (7,5) \text{ is a c.p.} \end{array}}$

2. Find the Jacobian:

$$\underline{\underline{J}} = \begin{bmatrix} 10 - 2y & -2x \\ 3y & -21 + 3x \end{bmatrix}$$

3. Evaluate $\underline{\underline{J}}$ at each c.p. and determine the type and stability

@ $(0,0)$ $\quad \underline{\underline{J}} = \begin{bmatrix} 10 & 0 \\ 0 & -21 \end{bmatrix}$

$\lambda^2 + 11\lambda - 210 = 0$

$\lambda_1 = 10 \quad \underline{v}^{(1)} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\lambda_2 = -21 \quad \underline{v}^{(2)} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

so $(0,0)$ is a <u>saddle pt</u> and <u>unstable</u>

@ (7,5) $\quad \underline{\underline{J}} = \begin{bmatrix} 10-2\cdot 5 & -2\cdot 7 \\ 3\cdot 5 & -21+3\cdot 7 \end{bmatrix} = \begin{bmatrix} 0 & -14 \\ 15 & 0 \end{bmatrix}$

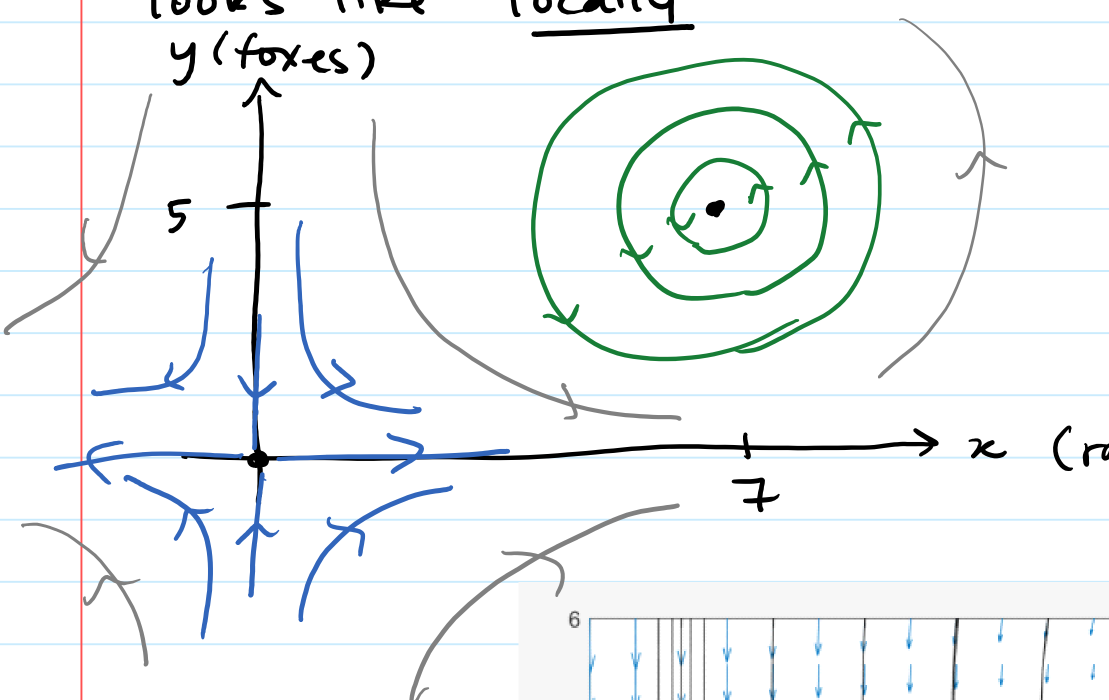
$D = 210 \qquad T = 0$

$\lambda^2 + 210 = 0$

$\lambda = \pm i\sqrt{210}$

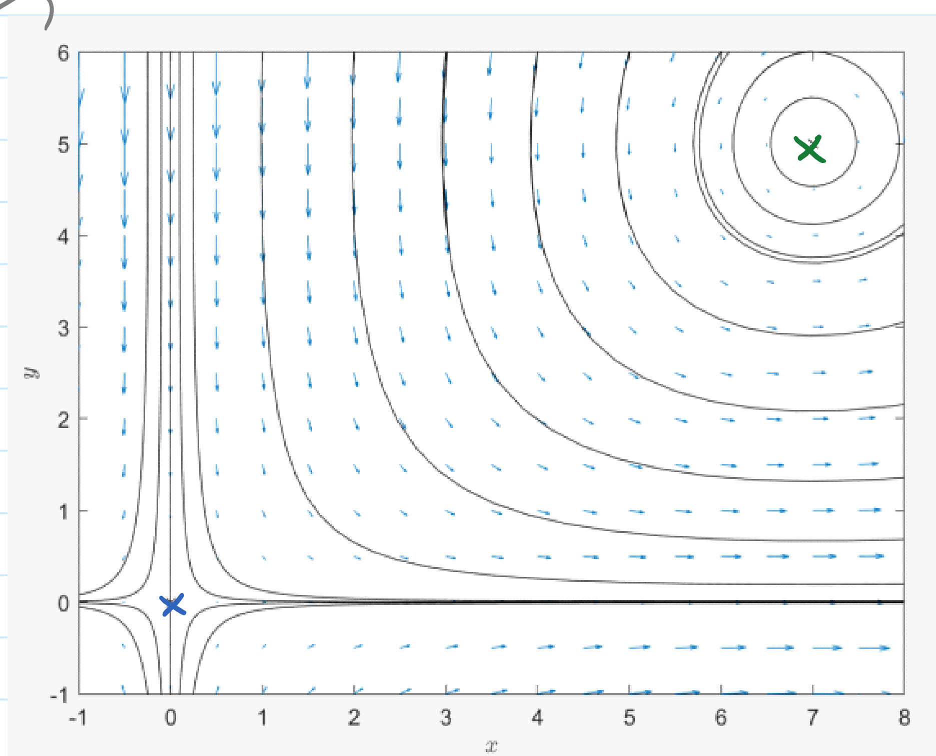
so (7,5) is a <u>center</u> and <u>stable</u>

So by linearizing the Predator-Prey model around the critical points (0,0) and (7,5), we can determine what the phase plane looks like <u>locally</u>




we can fill in the stream lines in between the c.p.

the above sketch matches the computed phase portrait →

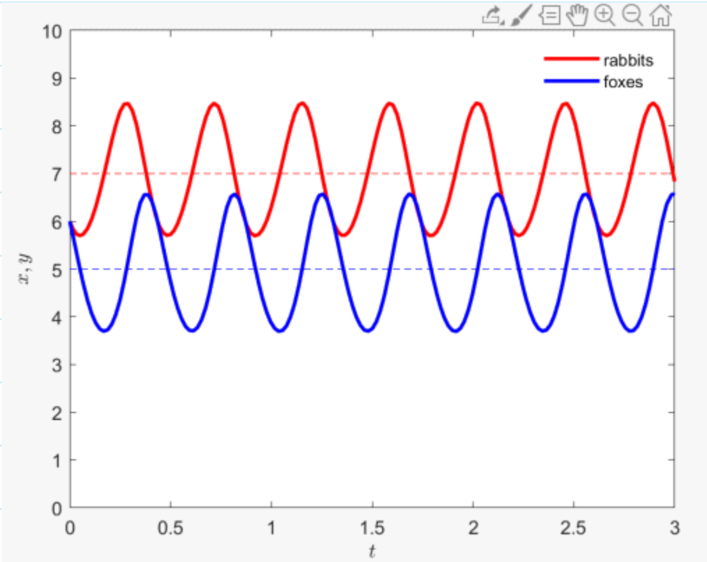


Q: What does this mean physically?

If $x_0 > 0$ and $y_0 > 0$, then as $t \to \infty$ the populations will stabilize around $(5,7)$

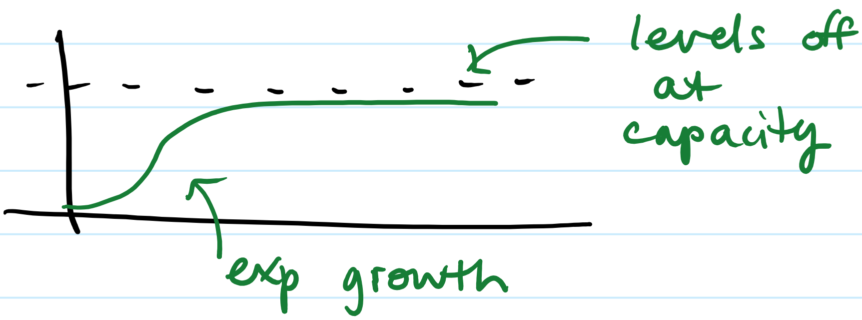
Because $(5,7)$ is a center, the rabbit population oscillates around $x=5$ (does not converge)

same for foxes and $y=7$.




Now, suppose the rabbits $x(t)$, instead of growing exponentially ($x' = ax$) grew logistically ($x' = ax - bx^2$)

Recall the logistic function




Q: How does logistic growth affect the predator–prey model?

<u>Ex:</u> $x' = 10x - \frac{1}{2}x^2 - 2xy$

$y' = -21y + 3xy$

1. Find the critical points:

$10x - \frac{1}{2}x^2 - 2xy = 0$ $\qquad$ $-21y + 3xy = 0$

$x(10 - \frac{1}{2}x - 2y) = 0$ $\qquad$ $y(-21 + 3x) = 0$

if $y = 0 \to x = 0$

$\qquad \searrow 10 - \frac{1}{2}x = 0$

if $x = 7 \to 10 - \frac{7}{2} - 2y = 0$

$$\boxed{\begin{array}{l} (0,0) \text{ is a c.p.} \\ (20,0) \text{ is a c.p.} \\ (7, \frac{13}{4}) \text{ is a c.p.} \end{array}}$$

2. Find the Jacobian:

$$\underline{\underline{J}} = \begin{bmatrix} 10 - x - 2y & -2x \\ 3y & -21 + 3x \end{bmatrix}$$

3. Evaluate the Jacobian at each c.p. and determine the type and stability

@ $(0,0)$ $\quad \underline{\underline{J}} = \begin{bmatrix} 10 & 0 \\ 0 & -21 \end{bmatrix}$ $\quad \lambda = -21, 10$ $\quad$ saddle pt unstable

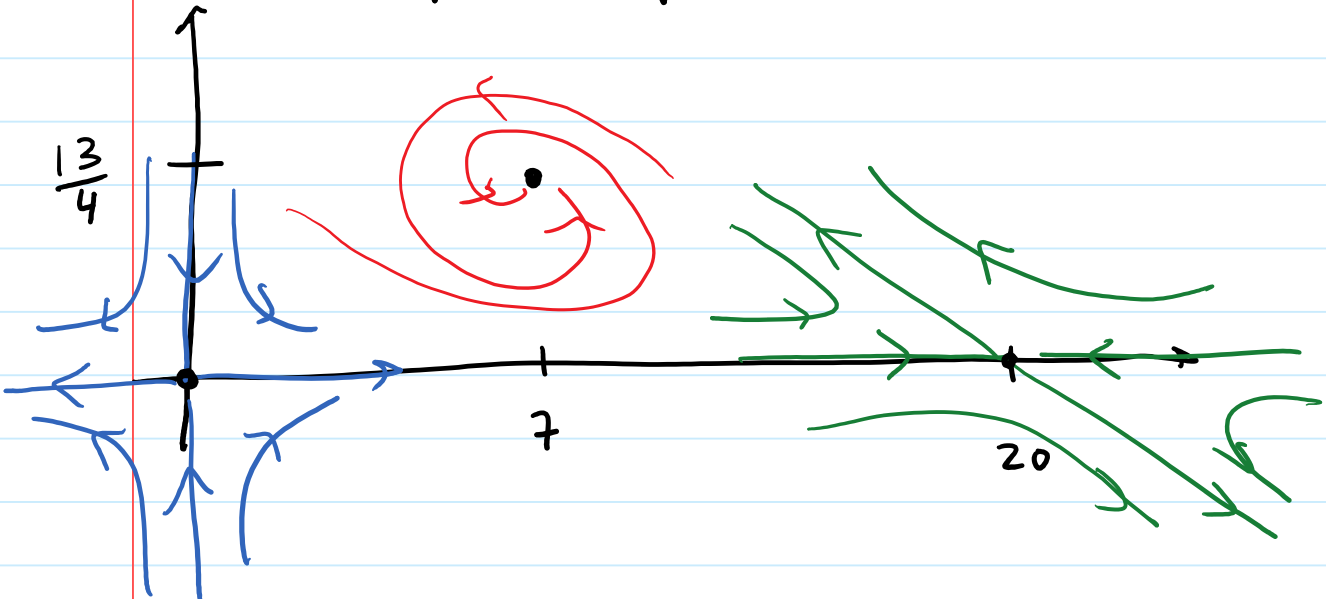
@ $(20,0)$ $\quad \underline{\underline{J}} = \begin{bmatrix} -10 & -40 \\ 0 & 39 \end{bmatrix}$ $\quad \lambda = 39, -10$ $\quad$ saddle pt unstable

@ $(7, \frac{13}{4})$ $\quad \underline{\underline{J}} = \begin{bmatrix} -7/2 & -14 \\ 39/4 & 0 \end{bmatrix}$ $\quad \lambda = -\frac{7}{4} \pm i\frac{\sqrt{2135}}{4}$

spiral sink

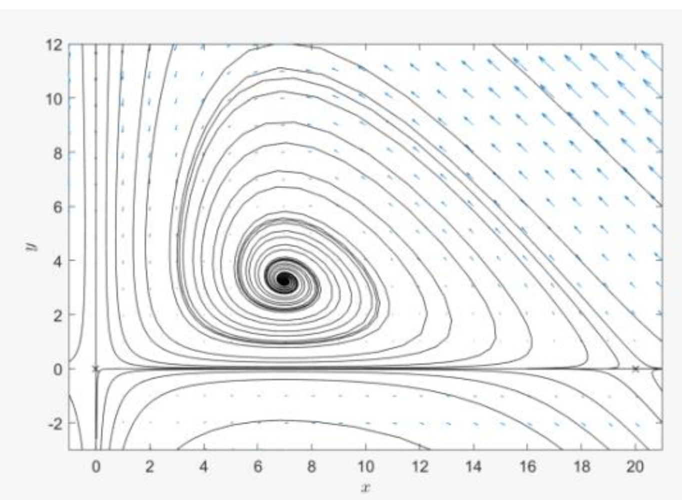
asymptotically stable

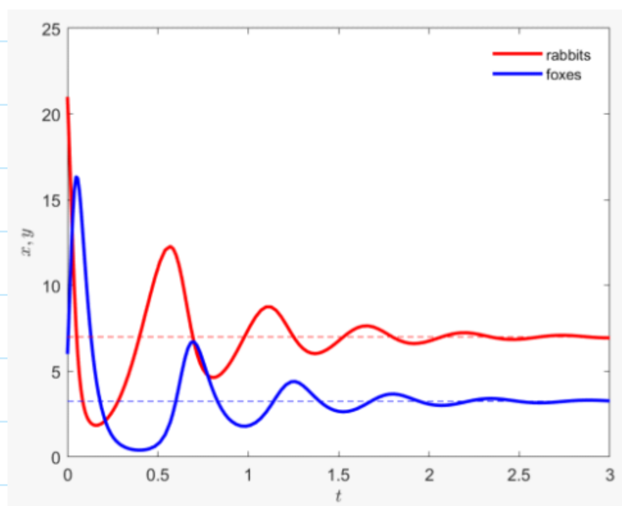
Sketch phase portrait




So as $t \to \infty$, $x \to 7$ (rabbits)
$y \to 13/4$ (foxes)

So the population of rabbits and foxes will converge to a single value



## II. Competitors:

Now consider two species that compete for food available in their common environment

$x(t)$ - rabbits $\qquad$ $y(t)$ - deer

- both eat vegetation
- neither preys on the other

The governing equations are:

$$\frac{dx}{dt} = \underbrace{\boxed{a_1 x - b_1 x^2}}_{\text{logistic growth}} - \underbrace{\boxed{c_1 xy}}_{\text{competition for resources}}$$

$$\frac{dy}{dt} = a_2 y - b_2 y^2 - c_2 xy$$

<u>Note</u>: $a_i, b_i, c_i > 0$

- the competition terms are <u>both</u> negative

(In Predator-Prey, competition terms have opposite signs)

Rewrite: $\quad x' = x\,(a_1 - b_1 x - c_1 y)$
$\qquad\qquad y' = y\,(a_2 - b_2 y - c_2 x)$

This system has <u>four</u> critical points

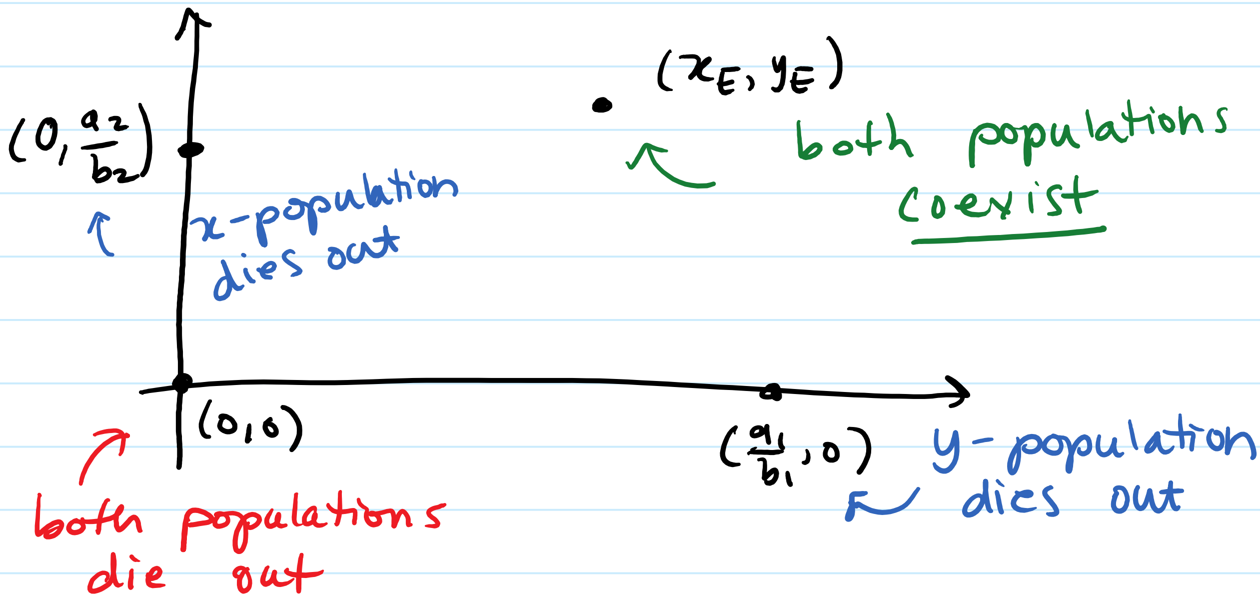
$(0,0)$ is a critical point

if $x=0 \to a_2 - b_2 y = 0$ $\qquad$ $\left(0, \frac{a_2}{b_2}\right)$ is a c.p.

if $y=0 \to a_1 - b_1 x = 0$ $\qquad$ $\left(\frac{a_1}{b_1}, 0\right)$ is a c.p.

The final critical point satisfies:

$$\left.\begin{aligned} a_1 - b_1 x - c_1 y &= 0 \\ a_2 - b_2 y - c_1 x &= 0 \end{aligned}\right\} \Rightarrow \text{Call solution } (x_E, y_E)$$




Often coexistence is a goal. So we want the c.p. $(x_E, y_E)$ to be stable

Coexistence Criteria:

If $\underbrace{c_1 c_2}_{\text{competition}} < \underbrace{b_1 b_2}_{\text{inhibition}}$ then $(x_E, y_E)$ is an asymptotically stable c.p.

So the two species coexist.

Ex: $x' = 30x - 3x^2 + xy = x(30 - 3x + y)$
$y' = 60y - 3y^2 + 4xy = y(60 - 3y + 4x)$

First, check the coexistence criteria
$b_1 = 3, \quad c_1 = 1 \qquad b_2 = 3 \quad c_2 = 4$

$$c_1 c_2 = 4 \quad < \quad 9 = b_1 b_2$$

So yes, the species should coexist.

Let's evaluate the phase plane:

1. Find critical points:
$(0,0)$ is a c.p. $\qquad (0,0)$ is a c.p.
if $x=0 \rightarrow 60 - 3y = 0 \qquad (0,20)$ is a c.p.
if $y=0 \rightarrow 30 - 3x = 0 \qquad (10,0)$ is a c.p.

the remaining c.p. solves:
$30 - 3x + y = 0 \longrightarrow y = 3x - 30$
$60 - 3y + 4x = 0$
$60 - 3(3x - 30) + 4x = 0$
$150 - 5x = 0 \longrightarrow x = 30$
$y = 3 \cdot 30 - 30 = 60 \qquad y = 60$

So the critical points are:
$(0,0) \qquad (0,20) \qquad (10,0) \qquad (30,60)$

2. Find the Jacobian:

$$\underline{\underline{J}} = \begin{bmatrix} 30-6x+y & x \\ 4y & 60-6y+4x \end{bmatrix}$$

3. Evaluate $\underline{\underline{J}}$ at each c.p. and determine the type + stability

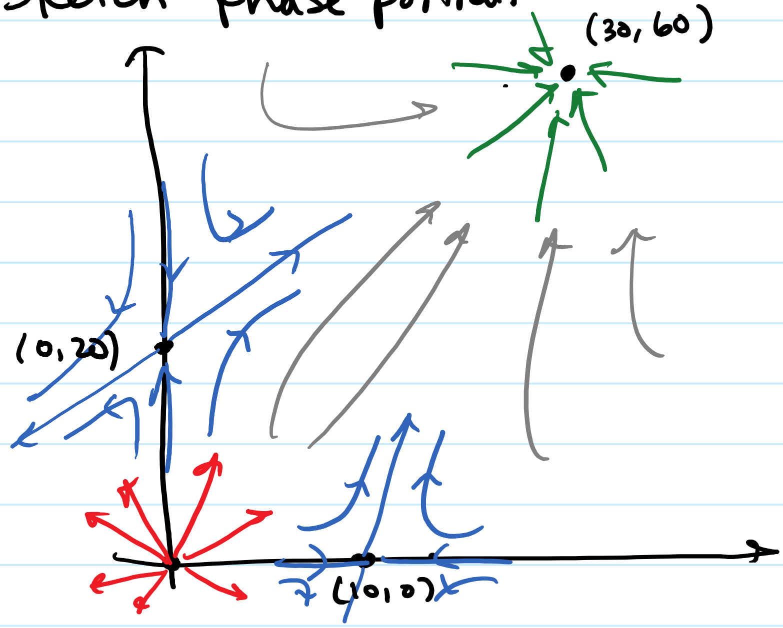
@ $(0,0)$ $\quad \underline{\underline{J}} = \begin{bmatrix} 30 & 0 \\ 0 & 60 \end{bmatrix}$ $\quad \lambda = 30, 60$ $\quad$ nodal source unstable

@ $(0,20)$ $\quad \underline{\underline{J}} = \begin{bmatrix} 50 & 0 \\ 80 & -60 \end{bmatrix}$ $\quad \lambda = -60, 50$ $\quad \underline{v} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 11 \\ 8 \end{bmatrix}$ $\quad$ saddle pt unstable

@ $(10,0)$ $\quad \underline{\underline{J}} = \begin{bmatrix} -30 & 10 \\ 0 & 100 \end{bmatrix}$ $\quad \lambda = 100, -30$ $\quad \underline{v} = \begin{bmatrix} 1 \\ 13 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\quad$ saddle pt unstable

@ $(30,60)$ $\quad \underline{\underline{J}} = \begin{bmatrix} -90 & 30 \\ 240 & -180 \end{bmatrix}$ $\quad \lambda = -15(9 \pm \sqrt{41})$ both $\lambda < 0$ $\quad$ nodal sink asymp. stable

Sketch phase portrait

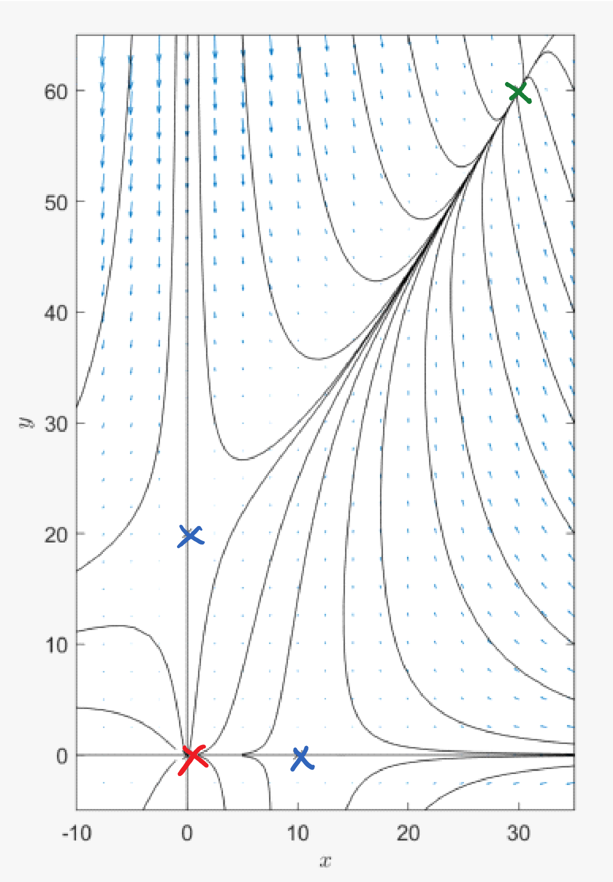


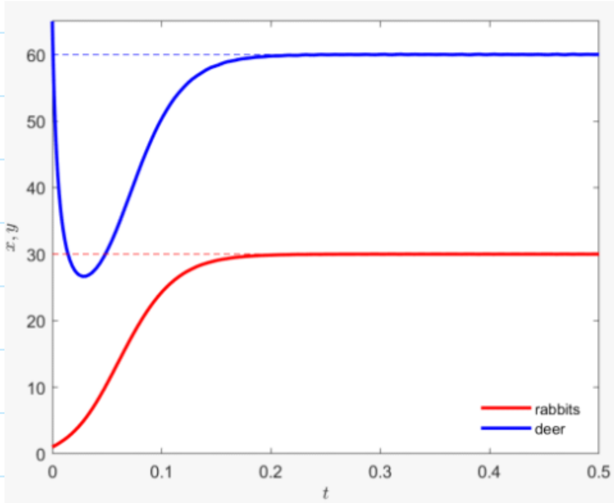

so as $t \to \infty$

$x \to 30$

$y \to 60$

the rabbit and deer populations will coexist




Computer phase portrait confirms our analysis.