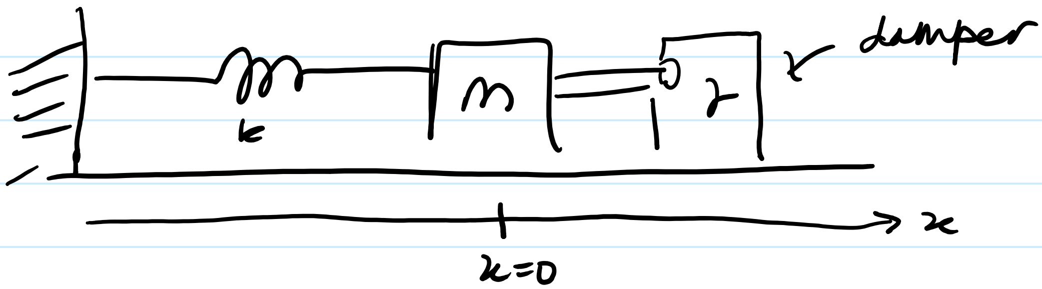
☆ Nonlinear Mechanical Systems:

I. Mass-Spring Damper System

Recall in MA 266, we studied mass-on-a-spring systems like:



The governing eqns were 2nd order linear:

$$m x'' + \gamma x' + kx = 0$$

Here $F_s = -kx$ is the spring force

Let's consider a nonlinear spring:

$$F_s = -kx + \beta x^3$$

"hard" if $\beta < 0$
"soft" if $\beta > 0$

Then $\quad m x'' + \gamma x' + kx - \beta x^3 = 0$

Convert this to a 1st order system:

$$\begin{cases} \dfrac{dx}{dt} = y \\ \dfrac{dy}{dt} = x'' = -\dfrac{\gamma}{m}x' - \dfrac{k}{m}x + \dfrac{\beta}{m}x^3 \end{cases}$$

Ex: "hard" spring $\quad m=1,\ \gamma=0,\ k=4,\ \beta=-9$

$$x'' + 4x + 9x^3 = 0$$

system: $\quad x' = y$
$\qquad\qquad\ y' = -4x - 9x^3$

1. Find critical points:

$y = 0$

$$-4x - 9x^3 = 0$$
$$-x\,(4 + 9x^2) = 0$$
$x = 0$ only real-valued solution

only critical pt: $(0,0)$

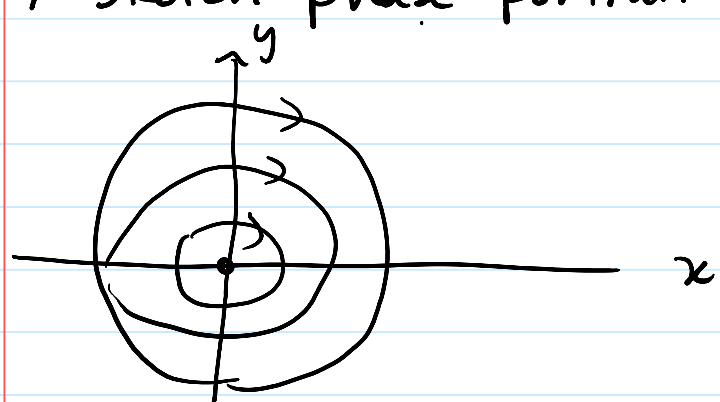
2. Find the Jacobian: $\underline{\underline{J}} = \begin{bmatrix} 0 & 1 \\ -4-27x^2 & 0 \end{bmatrix}$

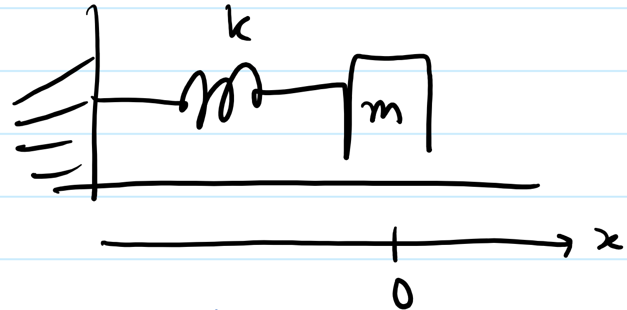
3. Evaluate $\underline{\underline{J}}$ at the critical point:

@ $(0,0)$ $\quad \underline{\underline{J}} = \begin{bmatrix} 0 & 1 \\ -4 & 0 \end{bmatrix}$ $\qquad \lambda^2 + 4 = 0$
$\qquad \lambda = \pm 2i$

<u>center</u> and <u>stable</u>

4. Sketch phase portrait:





mass oscillates around $x=0$ forever

Ex: "Soft" Spring $\qquad m=1,\ \gamma=0,\ k=4,\ \beta=+9$

$$x'' + 4x - 9x^3 = 0$$

Convert to system:

$$x' = y$$
$$y' = -4x + 9x^3$$

1. Find the critical points

$y=0$

$$-4x + 9x^3 = 0$$
$$x(-4+9x^2) = 0$$
$$x = 0 \qquad 9x^2 = 4$$
$$x = \pm\frac{2}{3}$$

There are 3 c.p.:

$(0,0)$, $\left(\frac{2}{3},0\right)$, $\left(-\frac{2}{3},0\right)$

2. Find the Jacobian: $\underline{\underline{J}} = \begin{bmatrix} 0 & 1 \\ -4+27x^2 & 0 \end{bmatrix}$

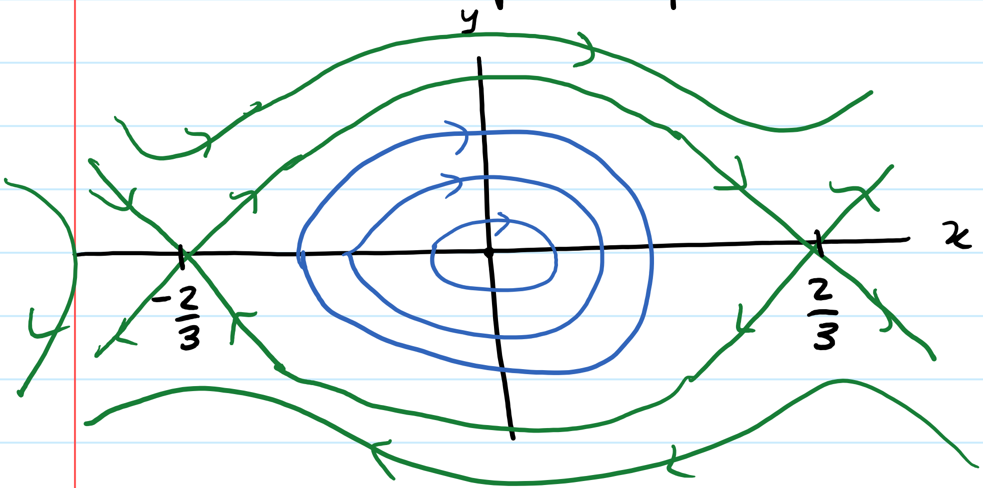
3. Evaluate $\underline{\underline{J}}$ at each critical point:

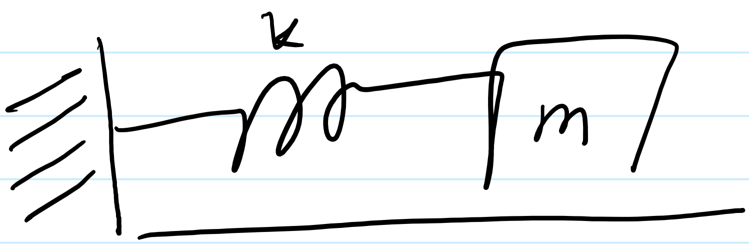
@ $(0,0)$ $\quad \underline{\underline{J}} = \begin{bmatrix} 0 & 1 \\ -4 & 0 \end{bmatrix}$ $\quad \lambda^2+4=0$, $\lambda = \pm 2i$ $\quad$ same as before: center, stable

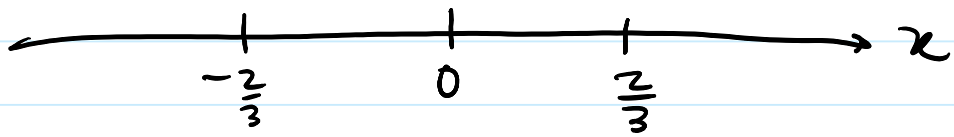
@ $\left(\frac{2}{3},0\right)$ $\quad \underline{\underline{J}} = \begin{bmatrix} 0 & 1 \\ -4+27\cdot\frac{4}{9} & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 8 & 0 \end{bmatrix}$ $\quad \lambda = \pm 2\sqrt{2}$ $\quad$ saddle pt, unstable

@ $\left(-\frac{2}{3},0\right)$ $\quad \underline{\underline{J}} = \begin{bmatrix} 0 & 1 \\ 8 & 0 \end{bmatrix}$ $\quad \lambda = \pm 2\sqrt{2}$ $\quad$ saddle pt, unstable

4. Sketch the phase portrait:







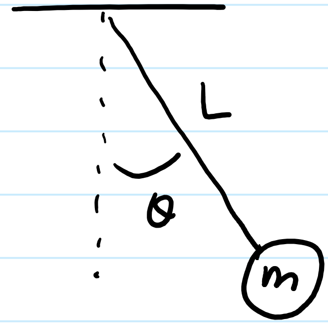
displacement grows unbdd

mass oscillates forever

displacement grows unbdd

mass initially heads toward $x=0$ but doesn't have enough energy to reach it, so it reverses direction.

II. Nonlinear Pendulum:



In MA 266, we derived the equations of motion for a pendulum:

$$\frac{d^2\theta}{dt^2} + \frac{g}{L}\sin\theta = 0$$

Then we made the small angle approximation

$$\sin\theta \approx \theta$$

$$\rightarrow \quad \theta'' + w^2\theta = 0 \qquad\qquad w^2 = g/L$$

$$\theta(t) = A\cos(wt) + B\sin(wt)$$

Now let's solve the fully nonlinear case:

$$\frac{d^2\theta}{dt^2} + \underbrace{c\frac{d\theta}{dt}}_{\text{damped motion}} + w^2\sin\theta = 0$$

(e.g. air resistance)

Convert to a system:

$$\begin{cases} x' = y \\ y' = -cy - w^2\sin(x) \end{cases}$$

Let's consider the undamped case $c=0$

$$\begin{cases} x' = y \\ y' = -w^2\sin(x) \end{cases}$$

$x' = y$

$y' = -w^2 \sin(x)$

1. Find the critical points:

$y = 0$ $\qquad$ $-w^2 \sin(x) = 0$

$\sin(x) = 0$ when $x = n\pi$, $n = 0, 1, 2 \ldots$

So the critical points are:

$(n\pi, 0)$, where $n = 0, 1, 2, 3 \ldots$

2. Find the Jacobian: $\underline{J} = \begin{bmatrix} 0 & 1 \\ -w^2\cos(x) & 0 \end{bmatrix}$

3. Evaluate $\underline{J}$ at each critical point:

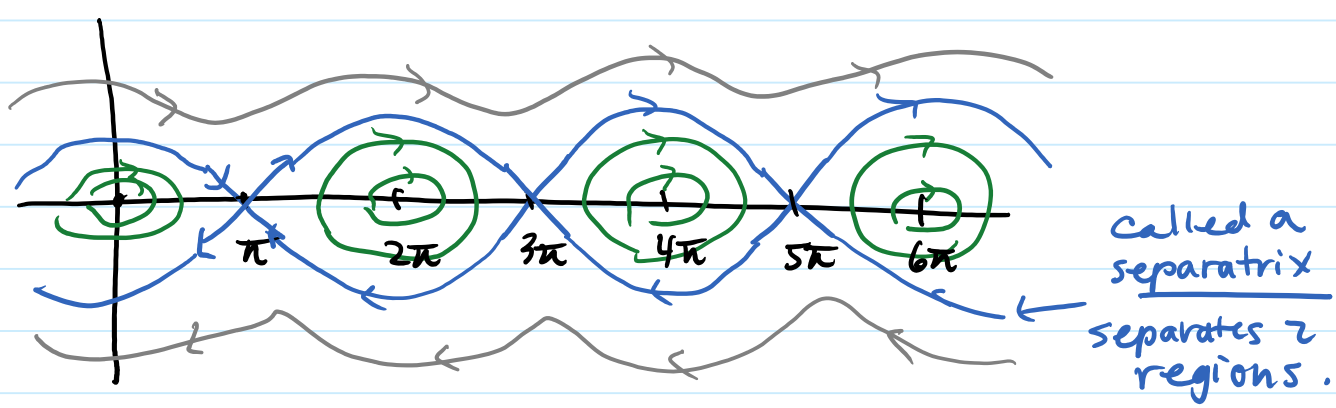
@ $(n\pi, 0)$, $n = 1, 3, 5 \ldots$ odd $\qquad \underline{J} = \begin{bmatrix} 0 & 1 \\ w^2 & 0 \end{bmatrix}$ $\qquad \lambda^2 - w^2 = 0$, $\lambda = \pm w$

saddle pt, unstable

@ $(n\pi, 0)$, $n = 0, 2, 4 \ldots$ even $\qquad \underline{J} = \begin{bmatrix} 0 & 1 \\ -w^2 & 0 \end{bmatrix}$ $\qquad \lambda^2 + w^2 = 0$, $\lambda = \pm iw$

center, stable

4. Sketch phase portrait:



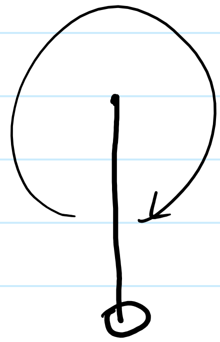
What does this mean physically for the nonlinear pendulum?

at $\theta = 0$

pendulum points straight down

obviously stable

at $\theta = 2\pi$



pendulum still points down

at $\theta = \pi$

Pendulum points up!

but <u>unstable</u>

any slight disturbance and it will fall to the downward position.