Lecture 9 (Sec 7.1)

☆ Chapter 7: Laplace Transform Methods

7.1: Laplace Transforms & Inverse Transforms

Let's go back to linear scalar ODE like:

$$mx'' + \gamma x' + kx = F(t)$$

mass on a spring

In MA 266, we were able to solve these ODE by guessing solutions like $x(t) = e^{rt}$ or using the forcing term $F(t)$ to inform an educated guess

In this chapter, we will use Laplace Transforms to solve <u>many</u> more types of problems.

I. Definition:

The <u>Laplace</u> <u>Transform</u> of a fcn $f(t)$ is:

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty \underbrace{e^{-st}}_{\text{"kernel" of the transform}} f(t)\, dt$$

<u>Note</u>: The kernel is an exponential fcn e^{-st}

In MA 266 we saw that e^t shows up in a lot of solutions of <u>linear</u> ODE, so this is a good choice for the kernel.

In this lecture, we will practice doing a <u>lot</u> of Laplace Transforms.
Next lecture we will see how to use it to solve ODE.

<u>Ex</u>: Find the L.T. of $f(t) = 1$

$$\mathcal{L}\{1\} = \int_0^\infty e^{-st}(1)\,dt = \int_0^\infty e^{-st}\,dt$$

indefinite integral

$$= \lim_{b\to\infty} \int_0^b e^{-st}\,dt$$

$$= \lim_{b\to\infty}\left[\frac{e^{-st}}{-s}\right]_0^b = \lim_{b\to\infty}\left[\frac{e^{-bs}}{-s} - \frac{1}{-s}\right]$$

this converges to 0 if $s>0$

$$\boxed{\mathcal{L}\{1\} = \frac{1}{s} \quad \text{if } s>0}$$

<u>Ex</u>: Find the L.T. of $f(t) = 3$

$$\mathcal{L}\{3\} = \int_0^\infty 3e^{-st}\,dt = 3\int_0^\infty e^{-st}\,dt$$

$$= 3\,\mathcal{L}\{1\} = \frac{3}{s}$$

so $\mathcal{L}\{3\} = \frac{3}{s}$ if $s>0$

So $\mathcal{L}\{a\,f(t)\} = a\,\mathcal{L}\{f(t)\}$

Ex: Find the L.T. of $f(t) = e^{at}$

$$\mathcal{L}\{e^{at}\} = \int_0^\infty e^{-st} e^{at} dt = \lim_{b\to\infty} \int_0^b e^{-(s-a)t} dt$$

$$= \lim_{b\to\infty} \left[\frac{e^{-(s-a)t}}{-(s-a)} \right]^b = \lim_{b\to\infty} \left[\underbrace{\frac{e^{-(s-a)b}}{-(s-a)}}_{\text{converges to } 0 \text{ if } s-a>0} - \frac{1}{-(s-a)} \right]$$

$$\boxed{\mathcal{L}\{e^{at}\} = \frac{1}{s-a} \quad \text{if } s>a}$$

II. Linearity:

Ex: Find the L.T. of $f(t) = 1 + e^{-3t}$

$$\mathcal{L}\{1 + e^{-3t}\} = \int_0^\infty e^{-st} (1 + e^{-3t}) dt$$

$$= \int_0^\infty e^{-st} dt + \int_0^\infty e^{-st} e^{-3t} dt$$

$$= \mathcal{L}\{1\} + \mathcal{L}\{e^{-3t}\}$$

$$= \frac{1}{s} + \frac{1}{s+3}$$

$$\boxed{\mathcal{L}\{1 + e^{-3t}\} = \frac{1}{s} + \frac{1}{s+3} \quad \text{if } s>0}$$

So the Laplace Transform satisfies:

$$\mathcal{L}\{af(t)\} = a\mathcal{L}\{f(t)\}$$

$$\mathcal{L}\{f(t) + g(t)\} = \mathcal{L}\{f(t)\} + \mathcal{L}\{g(t)\}$$

So the Laplace Transform is <u>linear</u>

(<u>Note</u>: this is <u>NOT</u> true for multiplication)

III. More Transforms

Ex: Find the L.T. of $f(t) = \cos(t)$

Recall that $\cos(t) = \frac{1}{2}\left(e^{it} + e^{-it}\right)$

Then

$$\mathcal{L}\{\cos(t)\} = \mathcal{L}\left\{\frac{1}{2}e^{it} + \frac{1}{2}e^{-it}\right\}$$

$$= \frac{1}{2}\mathcal{L}\{e^{it}\} + \frac{1}{2}\mathcal{L}\{e^{-it}\}$$

$$= \frac{1}{2}\underbrace{\left(\frac{1}{s-i}\right)}_{\text{if } s > Re(i) = 0} + \frac{1}{2}\underbrace{\left(\frac{1}{s+i}\right)}_{\text{if } s > Re(-i) = 0}$$

$$= \frac{1}{2}\left[\frac{1}{s-i}\frac{(s+i)}{(s+i)} + \frac{1}{s+i}\frac{(s-i)}{(s-i)}\right]$$

$$= \frac{1}{2}\left[\frac{s+\cancel{i}+s-\cancel{i}}{s^2+\cancel{is}-\cancel{is}+1}\right] = \frac{1}{2}\left(\frac{2s}{s^2+1}\right) = \frac{s}{s^2+1}$$

$$\boxed{\mathcal{L}\{\cos(t)\} = \frac{s}{s^2+1} \quad \text{if } s > 0}$$

Ex: Find the L.T. of $f(t) = 5t$

$$\mathcal{L}\{5t\} = \int_0^\infty e^{-st}(5t)\,dt = 5\lim_{b\to\infty}\int_0^b e^{-st}\,t\,dt$$

Use Integration By Parts $\quad u = t \quad dv = e^{-st}dt$

$du = dt \quad v = \dfrac{e^{-st}}{-s}$

$$= 5\lim_{b\to\infty}\left[\left(t\frac{e^{-st}}{-s}\right)_0^b - \int_0^\infty \frac{e^{-st}}{-s}\,dt\right]$$

$$= 5\lim_{b\to\infty}\left[\left(t\frac{e^{-st}}{-s}\right)_0^b - \left(\frac{e^{-st}}{s^2}\right)_0^b\right]$$

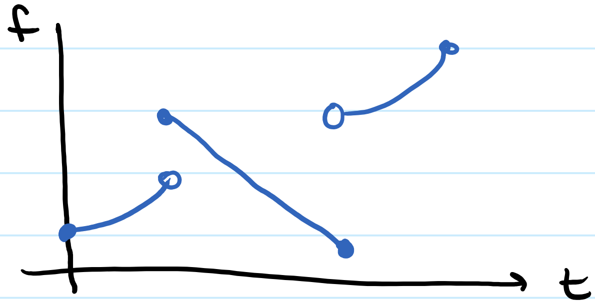
$$= 5\lim_{b\to\infty}\left[\underbrace{\frac{be^{-bs}}{-s}}_{\text{converges to 0 if } s>0} - \cancel{\frac{0e^0}{-s}} - \underbrace{\frac{e^{-sb}}{s^2}}_{\text{converges to 0 if } s>0} + \frac{1}{s^2}\right]$$

$$= 5\left(\frac{1}{s^2}\right)$$

$$\boxed{\text{So} \quad \mathcal{L}\{5t^2\} = \frac{5}{s^2} \quad \text{if } s>0}$$

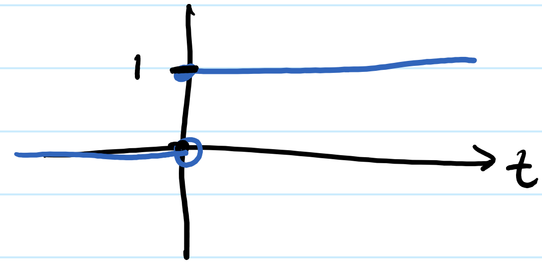
IV. Piecewise Linear Functions

Laplace Transforms also work for piecewise continuous functions



fcn has finitely many jumps.

Ex: $u(t) = \begin{cases} 0 & t<0 \\ 1 & t \geq 0 \end{cases}$



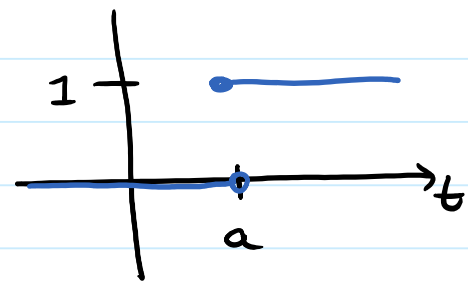
this is called the <u>unit step function</u>

$$\mathcal{L}\{u(t)\} = \int_0^\infty e^{-st}\, u(t)\,dt = \underbrace{\int_0^\infty e^{-st}\,dt}$$

because $u(t)=1$ on $t \geq 0$

$$= \mathcal{L}\{1\} = \frac{1}{s} \qquad \text{if } s>0$$

Ex: $u_a(t) = u(t-a) = \begin{cases} 0 & t<a \\ 1 & t \geq a \end{cases}$



$$\mathcal{L}\{u_a(t)\} = \int_0^\infty e^{-st}\, u_a(t)\,dt = \int_a^\infty e^{-st}\,dt$$

because $u_a(t)=0$ if $t<a$
$u_a(t)=1$ if $t \geq a$

$$= \lim_{b\to\infty} \int_a^b e^{-st}\,dt = \lim_{b\to\infty} \left[\frac{e^{-st}}{-s}\right]_a^b$$

$$= \lim_{b\to\infty} \left[\underbrace{\frac{e^{-sb}}{-s}} - \frac{e^{-sa}}{-s}\right] = \frac{e^{-as}}{s} \qquad \begin{matrix} s>0 \\ a>0 \end{matrix}$$

converges to 0
if $s>0$
and $a>0$

$$\boxed{\mathcal{L}\{u_a(t)\} = \frac{e^{-as}}{s} \quad \text{if } \begin{matrix} a>0 \\ s>0 \end{matrix}}$$

$$\boxed{\mathcal{L}\{u_a(t)\} = \frac{e^{-as}}{s} \quad \text{if} \quad \begin{matrix} a>0 \\ s>0 \end{matrix}}$$

V. Inverse Transform:

If $F(s) = \mathcal{L}\{f(t)\}$ then we call $f(t)$ the <u>inverse</u> <u>Laplace</u> <u>Transform</u> of $F(s)$

$$f(t) = \mathcal{L}^{-1}\{F(s)\}$$

<u>Ex</u>:

$$\mathcal{L}^{-1}\left\{\frac{1}{s+3}\right\} = e^{-3t}$$

$$\mathcal{L}^{-1}\left\{\frac{5}{s}\right\} = 5t$$

$$\mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\} = \cos(t)$$

In practice, we generally use a Table of Laplace Transforms to solve ODEs:

see table on next page

$f(t)$	$F(s)$	
1	$\frac{1}{s}$	$s>0$
t	$\frac{1}{s^2}$	$s>0$
$t^n \ (n\geq 0)$	$\frac{n!}{s^{n+1}}$	$s>0$
e^{at}	$\frac{1}{s-a}$	$s>a$
$\cos(kt)$	$\frac{s}{s^2+k^2}$	$s>0$
$\sin(kt)$	$\frac{k}{s^2+k^2}$	$s>0$
$\cosh(kt)$	$\frac{s}{s^2-k^2}$	$s>\lvert k\rvert$
$\sinh(kt)$	$\frac{k}{s^2-k^2}$	$s>\lvert k\rvert$
$u(t-a)$	$\frac{e^{-as}}{s}$	$s>0$ $a>0$

$\underline{\text{Ex}}$: $\mathcal{L}\{3e^{2t} + 2\sin^2(3t)\}$

$2\sin^2\theta = 1 - \cos(2\theta)$
$\theta = 3t$

$= \mathcal{L}\{3e^{2t} + 1 - \cos(6t)\}$

because $\mathcal{L}\{\}$ is linear

$= 3\mathcal{L}\{e^{2t}\} + \mathcal{L}\{1\} - \mathcal{L}\{\cos(6t)\}$

$= 3\left(\frac{1}{s-2}\right) + \frac{1}{s} - \frac{s}{s^2+36}$

$= \frac{3s^3 + 144s - 72}{s(s-2)(s^2+36)}$ ← after finding common denominator and simplifying