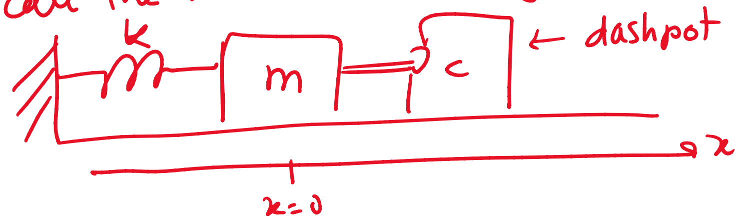
Lecture 10

Section 6.4:
Nonlinear Mechanical Systems

Announcements:
HW + A2 due Tuesday Jun 29
Midterm 1 on Thurs July 1

Warm up:

Recall the mass-on-a-spring problem:



$x(t)$ - displacement of the spring

Equation: $mx'' + cx' + kx = 0$

char eqn: $mr^2 + cr + k = 0$

$$r = \frac{-c}{2m} \pm \frac{\sqrt{c^2 - 4 \cdot m \cdot k}}{2m}$$

Fundamental solns

$x_1(t) = e^{r_1 t}$

$x_2(t) = e^{r_2 t}$

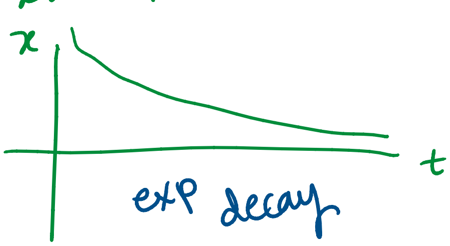
Sketch the solutions for the following cases:

over damped

$c^2 - 4mk > 0$

r_1, r_2 are real-valued negative

$x(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$

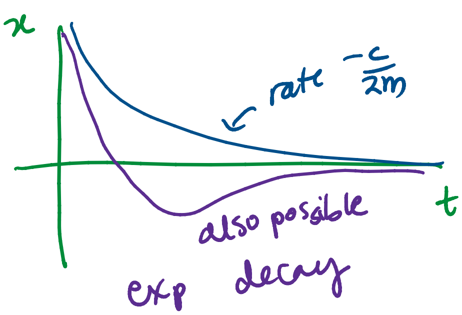


critically damped

$c^2 - 4mk = 0$

$r = \frac{-c}{2m}$ repeated root

$x(t) = c_1 e^{rt} + c_2 t e^{rt}$



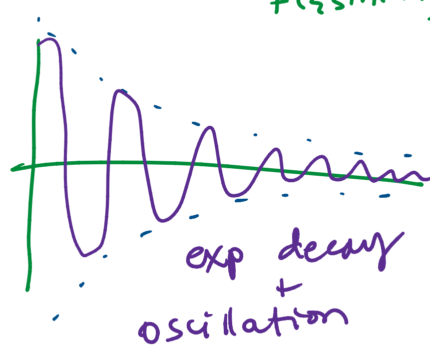
under damped

$c^2 - 4mk < 0$

$r = \frac{-c}{2m} \pm i\omega$

complex valued roots

$x(t) = e^{\frac{-ct}{2m}}\left(c_1 \cos(\omega t) + c_2 \sin(\omega t)\right)$



I. Mass-on-a-Spring system:

Linear case:

$mx'' + cx' + kx = 0$

$$\begin{cases} x' = y \\ y' = x'' = -\frac{c}{m} y - \frac{k}{m} x \end{cases}$$

Linear case:

$$mx'' + cx' + kx = 0$$

Convert to a 2D system: $\begin{cases} x' = y \\ y' = x'' = -\frac{c}{m}y - \frac{k}{m}x \end{cases}$

Matrix form: $\underline{x}' = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{c}{m} \end{bmatrix} \underline{x}$ linear system, c.p. at $(0,0)$

Look at eigenvalues λ

$T = \text{trace}(\underline{A}) = 0 - \frac{c}{m} = -\frac{c}{m}$

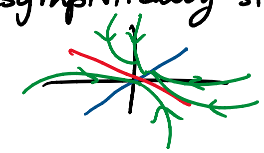
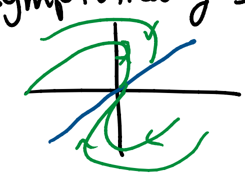
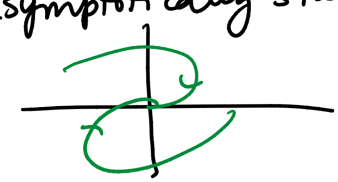
$D = \det(\underline{A}) = 0 + \frac{k}{m} = \frac{k}{m}$

$$\lambda^2 - T\lambda + D = 0$$

$$\lambda^2 + \frac{c}{m}\lambda + \frac{k}{m} = 0$$

$$\lambda = \frac{-c}{2m} \pm \frac{\sqrt{c^2 - 4mk}}{2m}$$

same char. eq. as for r

case	eigenvalues	phase portrait
$c^2 - 4mk > 0$	λ_1, λ_2 real, distinct negative	improper nodal sink asymptotically stable 
$c^2 - 4mk = 0$	$\lambda_1 = \lambda_2 = \frac{-c}{2m}$ mult. eigenvalue defective	improper nodal sink asymptotically stable 
$c^2 - 4mk < 0$	$\lambda = \frac{-c}{2m} \pm i\omega$ Complex valued	spiral sink asymptotically stable 

Nonlinear case:

Before spring force

$$F_s = -kx$$

Consider a nonlinear spring

$$F_s = -kx + \beta x^3$$

"hard" spring if $\beta < 0$

Consider a

$$F_s = -kx + \beta x^3$$

"hard" spring if $\beta < 0$
"soft" spring if $\beta > 0$

Equations: $mx'' + cx' + kx - \beta x^3 = 0$

Convert to a 2D system

$$\begin{cases} x' = y \\ y' = -\frac{c}{m} y - \frac{k}{m} x + \frac{\beta}{m} x^3 \end{cases}$$

<u>Ex:</u> "hard" spring $m = 1, \; c = 0, \; k = 4, \; \beta = -9$

$$x'' + 4x + 9x^3 = 0$$

system: $\begin{cases} x' = y \\ y' = -4x - 9x^3 \end{cases}$

1. Find the critical points

$y = 0$

$$-4x - 9x^3 = 0$$

$$-x(4 + 9x^2) = 0$$

$x = 0$ $\quad$ $\cancel{x^2 = -\frac{4}{9}}$ $\leftarrow$ complex valued soln

ignore

$\boxed{\text{c.p. is } (0,0)}$

2. Linearize around c.p. $\longrightarrow$ Jacobian

$$\underline{J} = \begin{bmatrix} F_x & F_y \\ G_x & G_y \end{bmatrix} = \left. \begin{bmatrix} 0 & 1 \\ -4 - 27x^2 & 0 \end{bmatrix} \right|_{(0,0)} = \begin{bmatrix} 0 & 1 \\ -4 & 0 \end{bmatrix}$$

associated linear system: $\underline{x}' = \begin{bmatrix} 0 & 1 \\ -4 & 0 \end{bmatrix} \underline{x}$

3. Evaluate the type + stability

$T = 0 \qquad D = 0 + 4 = 4$

$$\lambda^2 - T\lambda + D = 0$$

$$\lambda^2 + 4 = 0$$

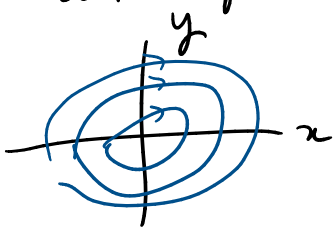
$$\lambda = \pm 2i$$

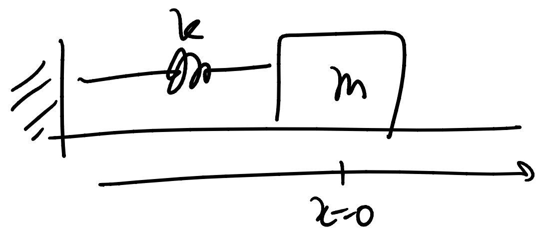
<u>center</u>
<u>stable</u>

$$\lambda^2 + 4 = 0$$
$$\lambda = \pm 2i$$

center
<u>Stable</u>

4. Sketch phase portrait





mass oscillates around $x=0$ forever.

<u>Ex</u>: "Soft" spring $\qquad m=1,\ c=0,\ k=4,\ \beta=+9$

$$\begin{cases} x' = y \\ y' = -4x + 9x^3 \end{cases}$$

1. Find the critical points

$y = 0$

$$-4x + 9x^3 = 0$$
$$x(9x^2 - 4) = 0$$
$x = 0 \qquad 9x^2 = 4$
$$x = \pm \tfrac{2}{3}$$

$$\boxed{\text{three critical points} \quad (0,0) \quad (\tfrac{2}{3},0) \quad (-\tfrac{2}{3},0)}$$

2. Linearize $\longrightarrow$ Find Jacobian

$$\underline{\underline{J}} = \begin{bmatrix} 0 & 1 \\ -4+27x^2 & 0 \end{bmatrix}$$

3. Evaluate the linear system at each c.p.

@ $(0,0)$ $\quad \underline{\underline{J}} = \begin{bmatrix} 0 & 1 \\ -4 & 0 \end{bmatrix}$ $\quad$ same as before $\lambda = \pm 2i$ $\quad$ center <u>stable</u>

@ $(\tfrac{2}{3}, 0)$ $\quad \underline{\underline{J}} = \begin{bmatrix} 0 & 1 \\ & \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ & \end{bmatrix}$ $\quad T = 0$, $D = -8$

@ $\left(\frac{2}{3}, 0\right)$ $\underline{\underline{J}} = \begin{bmatrix} 0 & 1 \\ -4 + 27 \cdot \frac{4}{9} & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 8 & 0 \end{bmatrix}$ $T = 0$, $D = -8$

$\lambda^2 - 8 = 0$

$\lambda = \pm\sqrt{8} = \pm 2\sqrt{2}$

saddle point
unstable

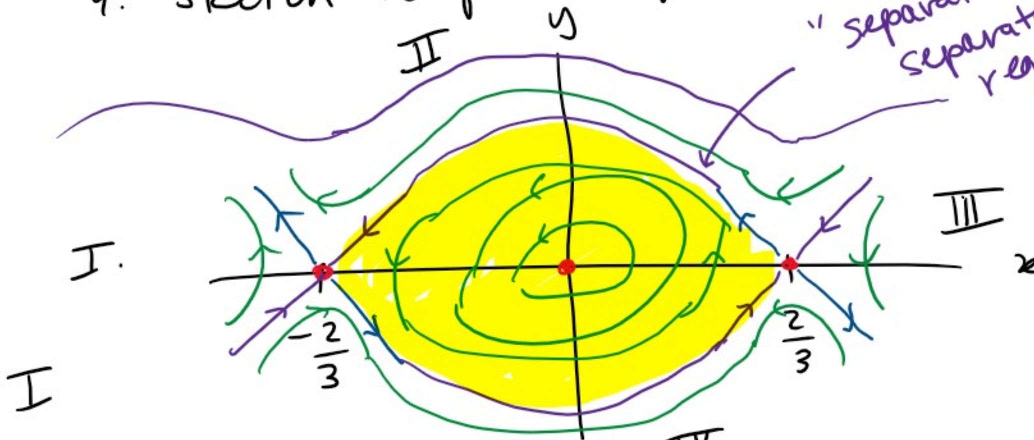
@ $\left(-\frac{2}{3}, 0\right)$ $\underline{\underline{J}} = \begin{bmatrix} 0 & 1 \\ 8 & 0 \end{bmatrix}$ same as $\left(\frac{2}{3}, 0\right)$

saddle point
unstable

$\lambda_1 = 2\sqrt{2}$ $\underline{v}^{(1)} = \begin{bmatrix} 1 \\ -2\sqrt{2} \end{bmatrix}$

$\lambda_2 = -2\sqrt{2}$ $\underline{v}^{(2)} = \begin{bmatrix} 1 \\ 2\sqrt{2} \end{bmatrix}$

4. Sketch a phase portrait



- solutions are stable

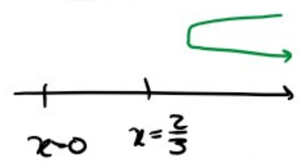
I

mass approaches $-\frac{2}{3}$, and turns away before reaching $-\frac{2}{3}$

b/c (0,0) is a center

b/c $\left(\frac{2}{3},0\right)$ is a saddle

III



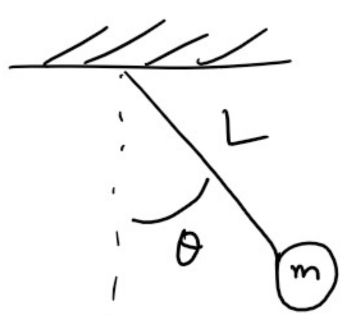
mass approaches $\frac{2}{3}$ and makes a u-turn

$F_s = -kx + \beta x^3$
$-4x + 9x^3$

II. Nonlinear Pendulum:

The equations of motion

$$\frac{d^2\theta}{dt^2} + \frac{g}{L}\sin\theta = 0$$



... the ...

$\frac{d^2\theta}{dt^2} \quad \frac{g}{L}$

In MA 266, we made the <u>small angle approximation</u>

if θ is small $\quad \sin\theta \approx \theta$

$$\rightarrow \quad \theta'' + \omega^2\theta = 0 \qquad \omega^2 = g/L$$

$$\theta(t) = A\cos(\omega t) + B\sin(\omega t)$$

Let's evaluate the nonlinear case

$$\underbrace{\theta'' + c\theta'}_{\text{damped motion}} + \omega^2 \sin\theta = 0$$

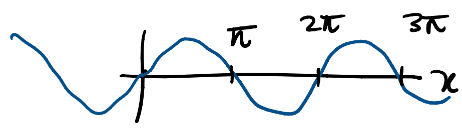
(e.g. air resistance)

Convert to a system of ODE's $\qquad x = \theta, \quad y = \theta'$

$$\begin{cases} x' = y \\ y' = -cy - \omega^2 \sin(x) \end{cases}$$

<u>Ex</u>: Let's analyze case when $c = 0$

$$\begin{cases} x' = y \\ y' = -\omega^2 \sin(x) \end{cases}$$



1. Find the critical points

$y = 0 \qquad -\omega^2 \sin(x) = 0$

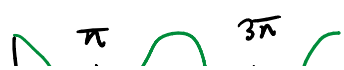
$x = n\pi$ where $n = 0, 1, 2, \ldots, -1, -2, -3$

$$\boxed{\text{infinitely many critical points } (n\pi, 0) \text{ where } n = \ldots -2, -1, 0, 1, 2, 3, \ldots}$$

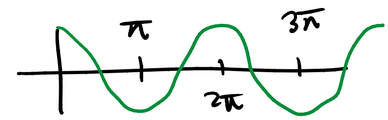
2. Linearize $\longrightarrow$ Find the Jacobian

$$J = \begin{bmatrix} F_x & F_y \\ G_x & G_y \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\omega^2\cos(x) & 0 \end{bmatrix}$$

... critical point



3. Evaluate $\underline{\underline{J}}$ at each critical point



@ $(n\pi, 0)$ $\quad \underline{\underline{J}} = \begin{bmatrix} 0 & 1 \\ -\omega^2 \cos(n\pi) & 0 \end{bmatrix}$

$\cos(n\pi) = +1$ if $n = 0, 2, 4, \ldots$

$\cos(n\pi) = -1$ if $n = 1, 3, 5, \ldots$

n odd
@ $(n\pi, 0)$
$n = 1, 3, 5, \ldots$

$\underline{\underline{J}} = \begin{bmatrix} 0 & 1 \\ \omega^2 & 0 \end{bmatrix}$ $\quad \lambda^2 - \omega^2 = 0$, $\lambda = \pm\omega$ $\quad$ saddle point, unstable

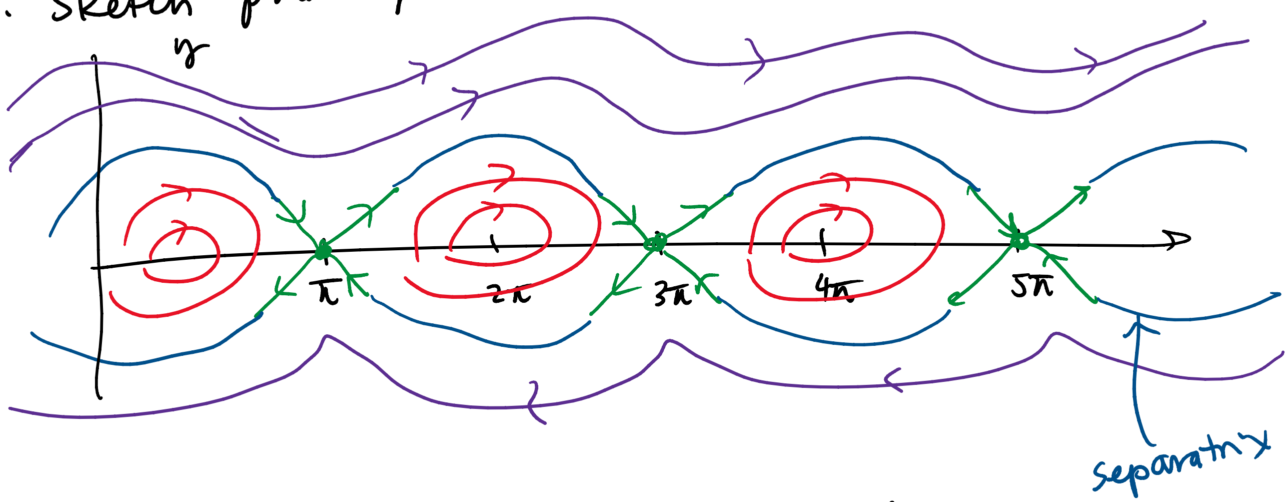
$T = 0, \quad D = -\omega^2$

n even
@ $(n\pi, 0)$
$n = 0, 2, 4, \ldots$

$\underline{\underline{J}} = \begin{bmatrix} 0 & 1 \\ -\omega^2 & 0 \end{bmatrix}$ $\quad \lambda^2 + \omega^2 = 0$, $\lambda = \pm i\omega$ $\quad$ center, stable

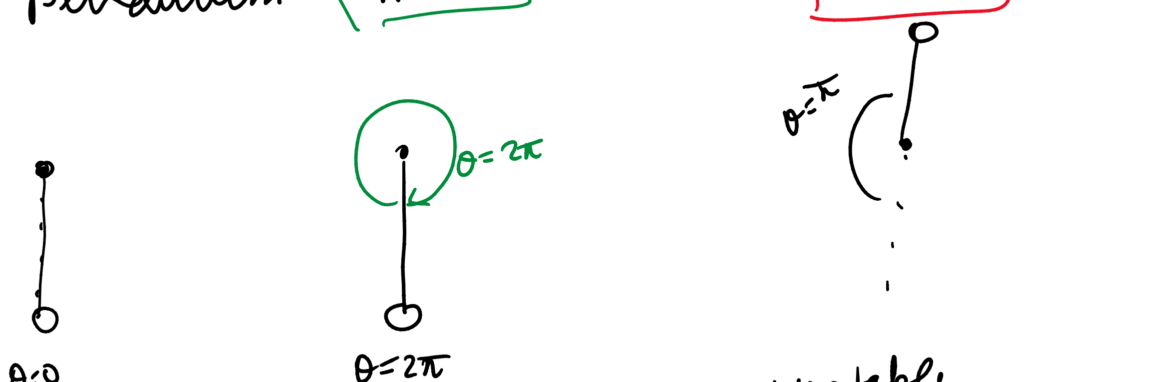
$T = 0 \quad D = \omega^2$

4. Sketch phase portrait



Q: What does this mean physically for the pendulum

n=even $\qquad$ n odd



$\theta = 0$

stable
equilibrium

$\theta = 2\pi$

stable
equilibrium

unstable
any slight disturbance
and the fall toward
$\theta = 0$