Lecture 11

★ Section 9.1:

Periodic Functions & Trigonometric Series

Announcements:

Online HW + A2 due Tuesday

Midterm 1 on Thursday July 1

Office Hours Today @ 2:30-3:30pm on Zoom

(new office: MATH 844)

Warm up:

Useful formulas for Ch 9

$$\int u\cos(u)\,du = \cos(u) - u\sin(u) + C$$

$$\int u\sin(u)\,du = \sin(u) - u\cos(u) + C$$

$$\int u^n\cos(u)\,du = u^n\sin(u) - n\int u^{n-1}\sin(u)\,du$$

$$\int u^n\sin(u)\,du = -u^n\cos(u) + n\int u^{n-1}\cos(u)\,du$$

★ all of these can be derived using integration by parts.

I. Periodic Functions:

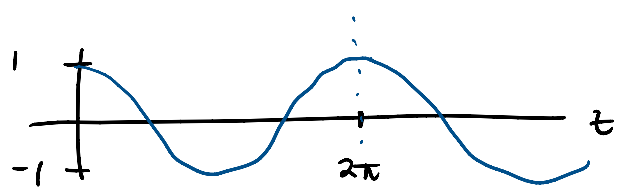
Def: A function $f(t)$ is periodic provided there exists $p>0$ such that

$$f(t+p) = f(t) \quad \text{for all } t$$

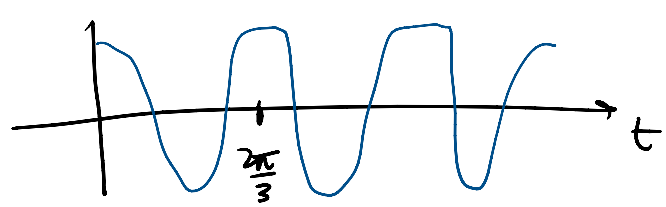
The value p is called the period of f

Ex: $f(t) = \cos(t)$

$p = 2\pi$



$f(t) = \cos(3t)$



$$\cos(3(t+p)) = \cos(3t+3p)$$
$$= \cos(3t)$$

$3p = 2\pi$ b/c cos has $p = 2\pi$

$p = \frac{2\pi}{3}$

$3p = 2\pi$ b/c cos has p ...

$p = \frac{2\pi}{3}$

<u>Ex</u>: $f(t) = 3 + \cos(t) - \sin(t) + 5\cos(2t)$

when $n = 1, 2, 3, \ldots$

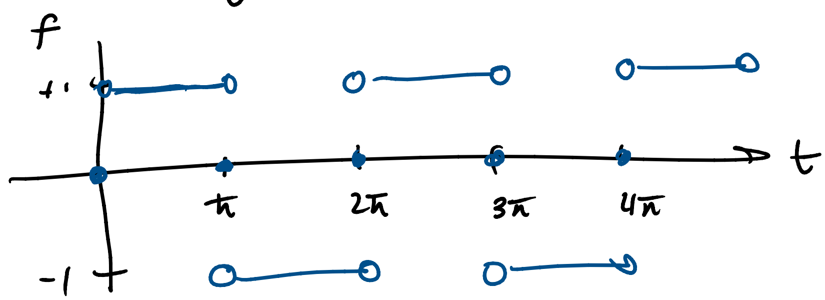
$f(t) = \cos(nt)$ then claim $p = 2\pi$ is a period of this function

$\cos(n(t+2\pi)) = \cos(nt + 2\pi n) = \cos(nt)$

$\sin(n(t+2\pi)) = \sin(nt + 2\pi n) = \sin(nt)$

So $f(t)$ also has period 2π

<u>Ex</u>: The square wave:



has period 2π

II, Fourier Series of period 2π functions:

Joseph Fourier (1822):
every function $f(t)$ with $p = 2\pi$ can be represented by an <u>infinite</u> trigonometric series

$$f(t) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos(nt) + b_n \sin(nt) \right)$$

a series of this form is called a <u>Fourier series</u>

this assertion is true with some mild restrictions on $f(t)$

GOAL: Derive the Fourier Series (F.S.) formulas

Need an important property of sines + cosines

Def: two functions $u(t)$ and $v(t)$ are orthogonal on the interval $[a,b]$ if

$$\int_a^b u(t)v(t)\,dt = 0$$

Observation: $\cos(nt)$ and $\sin(nt)$, for $n=1,2,3,\ldots$ are orthogonal on the interval $[-\pi,\pi]$

i.e. $$\int_{-\pi}^{\pi} \cos(mt)\sin(nt)\,dt = 0 \quad \text{for all } m \text{ and } n$$

$$\int_{-\pi}^{\pi} \cos(mt)\cos(nt)\,dt = \begin{cases} 0 & \text{if } n\neq m \\ \pi & \text{if } n=m \ * \end{cases}$$

$$\int_{-\pi}^{\pi} \sin(mt)\sin(nt)\,dt = \begin{cases} 0 & \text{if } n\neq m \\ \pi & \text{if } n=m \ * \end{cases}$$

* a function cannot be orthogonal to itself

Let's prove this for $\cos(mt)$ and $\cos(nt)$

$$\int_{-\pi}^{\pi} \underbrace{\cos(mt)\cos(nt)}_{\text{use a trig identity}}\,dt =$$

$$2\cos(A)\cos(B) = \cos(A-B) + \cos(A+B)$$

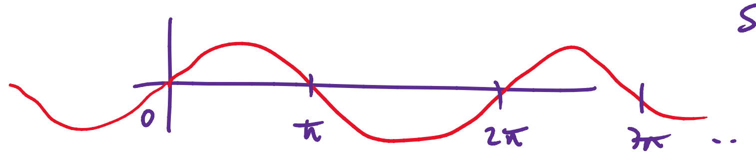
2 cases
$$\begin{cases} \text{if } m\neq n & \cos(mt)\cos(nt) = \frac{1}{2}\left[\cos((m-n)t) + \cos((m+n)t)\right] \\ \text{if } m=n & \cos(mt)\cos(mt) = \frac{1}{2}\left[\cancelto{1}{\cos(0)} + \cos(2mt)\right] \end{cases}$$

case $m\neq n$ $$= \int^{\pi} \frac{1}{2}\cos((m-n)t) + \frac{1}{2}\cos((m+n)t)\,dt$$

case $m \neq n$ $\quad = \displaystyle\int_{-\pi}^{\pi} \frac{1}{2}\cos((m-n)t) + \frac{1}{2}\cos((m+n)t)\,dt$

$$= \frac{1}{2}\left[\frac{\sin((m-n)t)}{m-n} + \frac{\sin((m+n)t)}{m+n} \right]_{-\pi}^{\pi}$$

$$= \frac{1}{2}\left[\frac{\cancelto{0}{\sin((m-n)\pi)}}{m-n} + \frac{\cancelto{0}{\sin((m+n)\pi)}}{m+n} - \frac{\cancelto{0}{\sin((m-n)(-\pi))}}{m-n} - \frac{\cancelto{0}{\sin((m+n)(-\pi))}}{m+n} \right]$$



$\sin(k\pi) = 0$ for all k integers

$k = 0, 1, 2, \ldots$
$-1, -2,$

So $\displaystyle\int_{-\pi}^{\pi} \cos(mt)\cos(nt)\,dt = 0$ if $n \neq m$

if $m = n$ $\quad = \displaystyle\frac{1}{2}\int_{-\pi}^{\pi} 1 + \cos(2mt)\,dt = \frac{1}{2}\left[t + \frac{\sin(2mt)}{2m} \right]_{-\pi}^{\pi}$

$$= \frac{1}{2}\left[\pi + \frac{\cancelto{0}{\sin(2m\pi)}}{m} - (-\pi) - \frac{\cancelto{0}{\sin(2m(-\pi))}}{2m} \right] = \pi$$

$\displaystyle\int_{-\pi}^{\pi} \cos(mt)\cos(mt)\,dt = \pi$ if $m = n$

NOTE: The other proofs are similar

Now, lets assume Fourier's assertion is true:

(*) $\displaystyle f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos(nt) + b_n\sin(nt)$

GOAL: Derive the values of a_n and b_n and a_0

$\boxed{\text{Case } a_0}$ Integrate both sides of (*) over $[-\pi, \pi]$

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$$\int_{-\pi}^{\pi} f(t)\,dt = \int_{-\pi}^{\pi} \frac{a_0}{2}\,dt + \int_{-\pi}^{\pi}\left(\sum_{n=1}^{\infty} a_n\cos(nt) + b_n\sin(nt)\right)dt$$

move integral inside the series

$$= \int_{-\pi}^{\pi}\frac{a_0}{2}dt + \sum_{n=1}^{\infty}\int_{-\pi}^{\pi} a_n\cos(nt) + b_n\sin(nt)\,dt$$

$$= \int_{-\pi}^{\pi}\frac{a_0}{2}dt + \sum_{n=1}^{\infty} a_n\int_{-\pi}^{\pi}\cos(nt)dt + \sum_{n=1}^{\infty} b_n\int_{-\pi}^{\pi}\sin(nt)dt$$

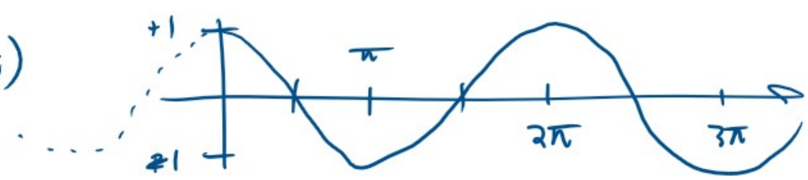
$$= \left[\frac{a_0 t}{2}\right]_{-\pi}^{\pi} + \sum_{n=1}^{\infty} a_n\left[\frac{\sin(nt)}{n}\right]_{-\pi}^{\pi} + \sum_{n=1}^{\infty} b_n\left[-\frac{\cos(nt)}{n}\right]_{-\pi}^{\pi}$$

$$= \left[\frac{a_0\pi}{2} - \frac{a_0(-\pi)}{2}\right] + \sum_{n=1}^{\infty} a_n\left[\cancelto{0}{\frac{\sin(n\pi)}{n}} - \cancelto{0}{\frac{\sin(-n\pi)}{n}}\right]$$

$$+ \sum_{n=1}^{\infty} b_n\left[-\frac{\cos(n\pi)}{n} + \left(+\frac{\overbrace{\cos(-n\pi)}^{=\cos(n\pi)}}{n}\right)\right]$$

$\cos(-x) = \cos(x)$

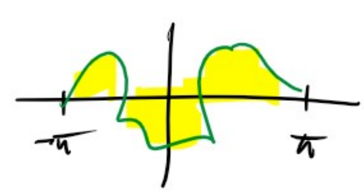
$\cos(m\pi)$



$$\cos(m\pi) = \left\{\begin{array}{ll} +1 & m = 0, 2, 4, \ldots \\ -1 & m = 1, 3, 5, \ldots \end{array}\right\} = (-1)^m$$

$$\int_{-\pi}^{\pi} f(t)\,dt = a_0\pi + \sum_{n=1}^{\infty} b_n\cancelto{0}{\left[-\frac{(-1)^n}{n} + \frac{(-1)^n}{n}\right]}$$

so $$\boxed{a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\,dt}$$



"average" value of $f(t)$

Think of this as the "average" value of $f(t)$ over $[-\pi,\pi]$

as we add more terms to the F.S., it will get closer to the form of $f(t)$.

$\boxed{\text{let's derive } a_n}$

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nt) + b_n \sin(nt)$$

multiply both sides by $\cos(mt)$ and integrate over $[-\pi,\pi]$

$$\int_{-\pi}^{\pi} f(t)\cos(mt)\,dt = \int_{-\pi}^{\pi} \frac{a_0}{2}\cos(mt)\,dt + \int_{-\pi}^{\pi} \cos(mt) \sum_{n=1}^{\infty} a_n \cos(nt) + b_n \sin(nt)\,dt$$

distribute integral

$$= \frac{a_0}{2}\underbrace{\int_{-\pi}^{\pi}\cos(mt)\,dt}_{0} + \sum_{n=1}^{\infty} a_n \int_{-\pi}^{\pi}\cos(mt)\cos(nt)\,dt = \begin{cases} 0 & \text{if } n \neq m \\ \pi & \text{if } n = m \end{cases}$$

$$+ \sum_{n=1}^{\infty} b_n \underbrace{\int_{-\pi}^{\pi}\cos(mt)\sin(nt)\,dt}_{0}$$

b/c sin and cos are orthogonal

$$= a_m \pi$$

$$\boxed{a_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(mt)\,dt}$$

Similarly, $\boxed{b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin(nt)\,dt}$

$$b_n = \frac{1}{\pi}\int_{-\pi} f(t)\sin(nt)\,dt$$

$\underline{\underline{\text{Def}}}$: If $f(t)$ is piecewise continuous with period 2π, then the <u>Fourier Series</u>

$$f(t) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos(nt) + b_n\sin(nt)$$

where

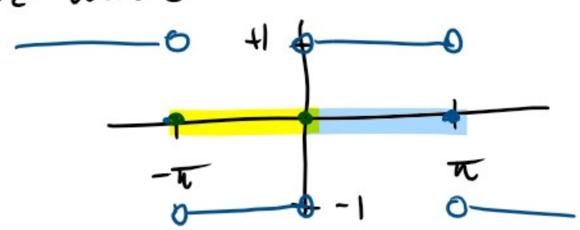
$$\left\{\begin{aligned} a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\,dt \\ a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(nt)\,dt \\ b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin(nt)\,dt \end{aligned}\right.$$

these are called the <u>Fourier coefficients</u>

<u>NOTE</u>: Sometimes this series fails to converge to $f(t)$ at some points t.
So we "$\sim$" instead "$=$" ← WHY? – Next class

<u>Ex</u>: Find the F.S. of the square wave

$$f(t) = \begin{cases} -1 & \text{if } -\pi < t < 0 \\ +1 & \text{if } 0 < t < \pi \\ 0 & \text{if } t = -\pi, 0, \pi \end{cases}$$



$$f(t) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos(nt) + b_n\sin(nt)$$

1. Calculate a_0

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\,dt = \frac{1}{\pi}\left[\int_{-\pi}^{0}(-1)\,dt + \int_{0}^{\pi}(+1)\,dt\right]$$

$$= \frac{1}{\pi}\left[(-t)\Big|_{-\pi}^{0} + (t)\Big|_{0}^{\pi}\right]$$

$$= \frac{1}{\pi}\left[(-t)\Big|_{-\pi}^{0} + (t)\Big|_{0}^{\pi} \right]$$

$$= \frac{1}{\pi}\left[-0 - (+(+\pi)) + (\pi - 0) \right] = 0$$

$$\boxed{a_0 = 0}$$

average value of f(t) is zero ✓

2. Calculate a_n

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(nt)\,dt =$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} (-1)\cos(nt)\,dt + \int_{0}^{\pi} (+1)\cos(nt)\,dt \right]$$

= calculate out = 0

3. Calculate b_n

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin(nt)\,dt$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} (-1)\sin(nt)\,dt + \int_{0}^{\pi} (+1)\sin(nt)\,dt \right]$$

$$= \frac{1}{\pi}\left[\left(\frac{\cos(nt)}{n} \right)\Big|_{-\pi}^{0} + \left(-\frac{\cos(nt)}{n} \right)\Big|_{0}^{\pi} \right]$$

$$= \frac{1}{\pi}\left[\frac{\cos(0)}{n} \xrightarrow{} 1 - \frac{\cos(-n\pi)}{n} \xrightarrow{} (-1)^n - \frac{\cos(n\pi)}{n} \xrightarrow{} (-1)^n + \frac{\cos(0)}{n} \xrightarrow{} 1 \right]$$

$$= \frac{1}{n\pi}\left[2 - 2(-1)^n \right] = \frac{2}{n\pi}\left[1 - (-1)^n \right]$$

$$\boxed{b_n = \begin{cases} \frac{4}{n\pi} & \text{if } n \text{ odd} \\ 0 & \text{if } n \text{ even} \end{cases}}$$

$$\boxed{b_n = \begin{cases} \frac{4}{n\pi} & \text{if } n \text{ odd} \\ 0 & \text{if } n \text{ even} \end{cases}}$$

Fourier series of the square wave:

$$\boxed{f(t) \sim \sum_{n \text{ odd}} \frac{4}{n\pi} \sin(nt)}$$

$\frac{4}{\pi}\sin(t)$

$\frac{4}{\pi}\sin(t) + \frac{4}{3\pi}\sin(3t)$