Lecture 12

Announcements:

- HW + A2 due Today @ 11:59pm
- Midterm 1 Thurs July 1st
- Review Lecture tomorrow
- Office Hours Today @ 2:30-3:30pm
- Lecture Friday will be a recorded video – no in class lecture → uploaded to BS

☆ Sec 9.2: General Fourier Series and Convergence

Warm up: From last class

Def: If the function $f(t)$ is

1) piecewise continuous and
2) periodic with $p = 2\pi$

then the Fourier series of $f(t)$ is

$$f(t) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nt) + b_n \sin(nt)$$

Write down the definitions of a_0, a_n, and b_n.

Fourier Coefficients $\quad a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\,dt$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(nt)\,dt \qquad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin(nt)\,dt$$

I. General Fourier Series:

Q: What if $f(t)$ is periodic but $p \neq 2\pi$?

A: We can adjust our definition to work for any periodic fcn.

Assume $f(t)$ has period $p = 2L$

Here L is called the half-period

☆ **Def:** the Fourier series of a $2L$-periodic function $f(t)$ is:

$$f(t) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\left(\frac{n\pi t}{L}\right) + b_n \sin\left(\frac{n\pi t}{L}\right)\right)$$

$$f(t) \sim \frac{a_0}{2} + \sum_{n=1} \left(a_n \cos\left(\frac{n\pi t}{L}\right) + b_n \sin\left(\frac{n\pi t}{L}\right)\right)$$

where the <u>Fourier</u> <u>coefficients</u> are

$$a_n = \frac{1}{L}\int_{-L}^{L} f(t)\cos\left(\frac{n\pi t}{L}\right)dt, \quad a_0 = \frac{1}{L}\int_{-L}^{L} f(t)\,dt$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(t)\sin\left(\frac{n\pi t}{L}\right)dt$$

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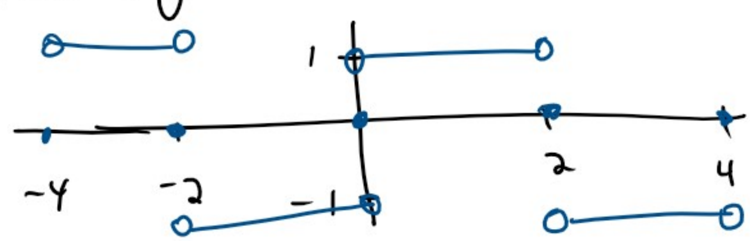
<u>NOTE</u>: From now on, let's use this definition.
works in <u>all</u> cases

previous definition only works if $p = 2\pi$

<u>Ex</u>: Find the F.S. of the square wave with $P=4$

$$f(t) = \begin{cases} -1 & -2 < t < 0 \\ +1 & 0 < t < 2 \\ 0 & t = -2, 0, 2 \end{cases}$$



If $P = 4$ $\quad \boxed{L = 2}$

$$f(t) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi t}{2}\right) + b_n \sin\left(\frac{n\pi t}{2}\right)$$

1. Calculate a_0

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(t)\,dt = \frac{1}{2}\int_{-2}^{2} f(t)\,dt$$

$$= \frac{1}{2}\left[\int_{-2}^{0}(-1)\,dt + \int_{0}^{2}(+1)\,dt\right]$$

$$= \frac{1}{2}\left[\left[-t\right]_{-2}^{0} + \left[t\right]_{0}^{2}\right] = \frac{1}{2}\left[0 - (+(+2)) + 2 - 0\right]$$

$$= \frac{1}{2}\left[\left[-t\right]_{-2} + \left(t\right)_0 \right] = \frac{1}{2}\left(0 - (+(+2)) + 2 - 0\right)$$

$$\boxed{a_0 = 0}$$

2. Calculate a_n

$$a_n = \frac{1}{L}\int_{-L}^{L} f(t)\cos\left(\frac{n\pi t}{L}\right)dt = \frac{1}{2}\int_{-2}^{2} f(t)\cos\left(\frac{n\pi t}{2}\right)dt$$

... do this calculation

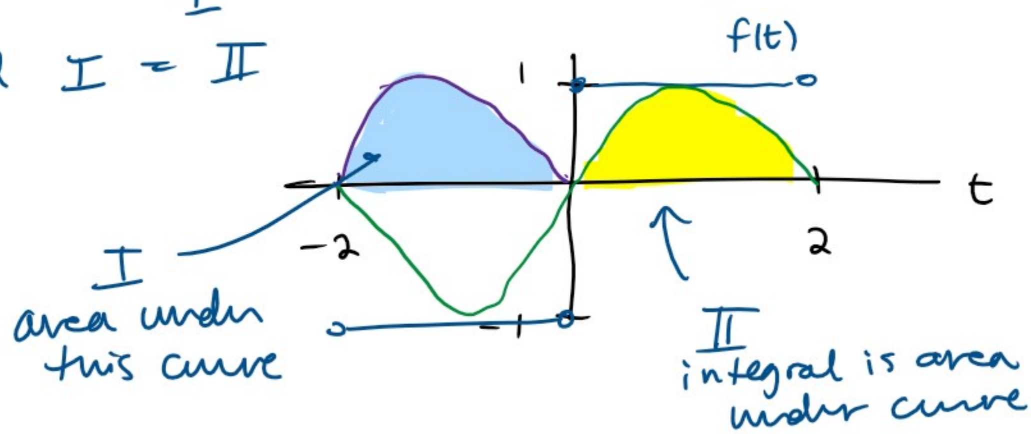
$$\boxed{a_n = 0}$$

3. Calculate b_n

$$b_n = \frac{1}{2}\int_{-2}^{2} f(t)\sin\left(\frac{n\pi t}{2}\right)dt$$

$$= \frac{1}{2}\left[\underbrace{\int_{-2}^{0}(-1)\sin\left(\frac{n\pi t}{2}\right)dt}_{\mathrm{I}} + \underbrace{\int_{0}^{2}(+1)\sin\left(\frac{n\pi t}{2}\right)dt}_{\mathrm{II}} \right]$$

Claim:
integral I = II



can also show this by making u-substitution $u = -t$ in integral I.

By symmetry:

$$= \frac{1}{\cancel{2}}\left[\cancel{2}\int_{0}^{2}(+1)\sin\left(\frac{n\pi t}{2}\right)dt\right]$$

$$= \left[-\cos\left(\frac{n\pi t}{2}\right)\right]^{2} = \frac{-2}{n\pi}\left[\overset{(-1)^n}{\cos\left(\frac{\cancel{2}n\pi}{\cancel{2}}\right)} - \overset{1}{\cos(0)}\right]$$

$$= \left[\frac{-\cos\left(\frac{n\pi t}{2}\right)}{\frac{n\pi}{2}} \right]_0^2 = \frac{-2}{n\pi}\left[\cos\left(\cancel{2}\frac{n\pi}{\cancel{2}}\right) - \cos(0)\right]$$

$$= \frac{-2}{n\pi}\left[(-1)^n - 1\right] = \boxed{\begin{cases} \frac{-4}{n\pi} & n \text{ odd} \\ 0 & n \text{ even} \end{cases} = b_n}$$

The Fourier Series is:

$$f(t) \sim \frac{\cancelto{0}{a_0}}{2} + \sum_{n=1}^{\infty} \cancelto{0}{a_n \cos\left(\frac{n\pi t}{2}\right)} + b_n \sin\left(\frac{n\pi t}{2}\right)$$

$$\boxed{f(t) \sim \sum_{n \text{ odd}} \frac{-4}{n\pi} \sin\left(\frac{n\pi t}{2}\right)}$$

Q: $f(t)$ square wave $p = 2\pi$

$f(t)$ square wave $p = 4$

(optional)

$$2\pi \xrightarrow{g} 4$$

$$g(2\pi) \longrightarrow 4$$

$$g(t) = \frac{4t}{2\pi} = \frac{2t}{\pi}$$

$$f_{2\pi}(t) = f_{2\pi}(g(t)) = f_4(t)$$

II. Convergence:

Last class — sometimes there are points t for which the F.S. does NOT converge

Q: When does the Fourier Series Converge?

Q! When does the Fourier Series Converge?

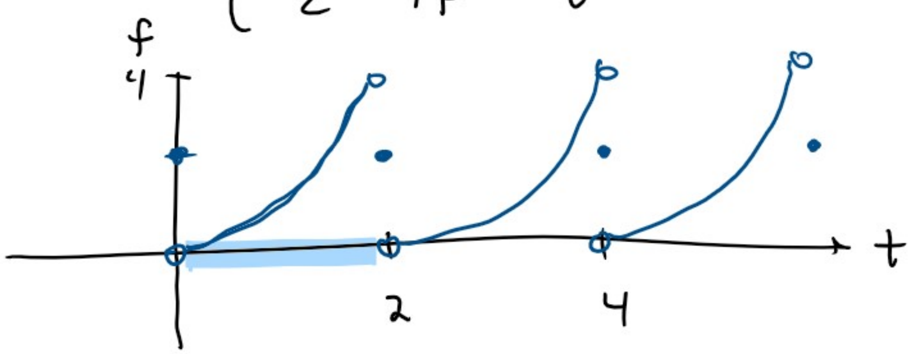
Thm: (Converge of Fourier Series)
If $f(t)$ is a periodic function that is piecewise smooth, then its Fourier series converges to:

(a) the value $f(t)$ at each point where f is continuous

(b) the value $\frac{1}{2}\left[f(t^+) + f(t^-)\right]$ at each point t where f is discontinuous

Ex: Let $f(t)$ have $P=2$ be defined by

$$f(t) = \begin{cases} t^2 & \text{if } 0<t<2 \\ 2 & \text{if } t=0,2 \end{cases}$$



when $0<t<2$, $f(t)$ is continuous
by Thm part (a), the F.S. will converge to $f(t)$

$$\text{F.S.} \xrightarrow[\text{on } 0<t<2]{} f(t)$$

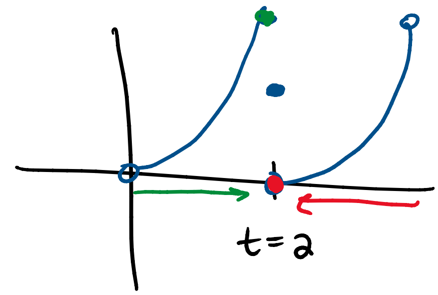
when $t=0, 2, 4, \ldots$, $f(t)$ is discontinuous
by Thm part (b)

$$\text{F.S.} \longrightarrow \frac{1}{2}\left[f(t^-) + f(t^+)\right]$$

by Thm part (b)

$$F.S. \longrightarrow \frac{1}{2}\left[f(t^-) + f(t^+)\right]$$

@ $t=2$



$$f(t^-) = \lim_{t\to 2^-} f(t) = \lim_{t\to 2^-} t^2 = 4$$

$$f(t^+) = \lim_{t\to 2^+} f(t) = \lim_{t\to 0^+} t^2 = 0$$

$$F.S.(@t=2) \longrightarrow \frac{1}{2}\left[f(t^-) + f(t^+)\right] = \frac{1}{2}\left[4+0\right] = 2$$

we defined the function $f(t)$ so that @ $t=2$ $\quad f(t) = \frac{1}{2}\left[f(t^+) + f(t^-)\right]$

so now, in this case, the F.S. converges for all t

Now, let's find the F.S. of $f(t)$

$$f(t) = \begin{cases} t^2 & 0<t<2 \\ 2 & t=0,2 \end{cases} \qquad \begin{matrix} P=2 \\ L=1 \end{matrix}$$

Fourier Series

By the Thm, we know that the F.S. converges for all t

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi t}{1}\right) + b_n \sin\left(\frac{n\pi t}{1}\right)$$

1. Calculate a_0

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(t)\,dt = \int_{-1}^{1} f(t)\,dt = \int_0^2 f(t)\,dt$$

because $f(t)$ is periodic w/ $P=2$

because $f(t)$ is periodic w/ $P=2$
we can use any interval with length P

$$= \int_0^2 t^2\,dt = \left[\frac{t^3}{3}\right]_0^2 = \frac{8}{3} - 0 = \boxed{\frac{8}{3} = a_0}$$

2. Calculate the a_n

$$a_n = \frac{1}{1}\int_0^2 f(t)\cos(n\pi t)\,dt = \int_0^2 t^2 \cos(n\pi t)\,dt$$

u substitution
$u = n\pi t$ $\quad t = \frac{u}{n\pi}$
$dt = \frac{du}{n\pi}$ $\quad t^2 = \frac{u^2}{(n\pi)^2}$

$$= \frac{1}{(n\pi)^3}\int_0^{2n\pi} u^2 \cos(u)\,du$$

use IBP or use formula

$$\int u^n \cos(u)\,du = u^n \sin(u) - n\int u \sin(u)\,du$$

$n=2$
$$\int u^2 \cos(u)\,du = u^2 \sin(u) - 2\int u\sin(u)\,du$$

$$= \frac{1}{(n\pi)^3}\left[\left(u^2\sin(u)\right)_0^{2n\pi} - \underbrace{2\int_0^{2n\pi} u\sin(u)\,du}_{\text{use another formula here}}\right]$$

$$\int u\sin(u)\,du = \sin(u) - u\cos(u) + C$$

$$= \frac{1}{(n\pi)^3}\left[u^2\sin(u) + 2\left(-\sin(u) + u\cos(u)\right)\right]_0^{2n\pi}$$

$$= \frac{1}{(n\pi)^3}\left[(2n\pi)^2\cancelto{0}{\sin(2n\pi)} - 2\cancelto{0}{\sin(2n\pi)} + 2(2n\pi)\cos(\cancelto{(-1)^{2n}}{2n\pi}) - 0^2\cancelto{0}{\sin(0)} + 2\cancelto{0}{\sin(0)} - 2\cancelto{0}{0\cdot\cos(0)}\right]$$

$$- 0^2\cancel{\sin(0)} + 2\cancel{\sin(0)} - 2\cancel{0}\cos(0) \;]$$

$$= \frac{1}{(n\pi)^{\cancel{3}}}\, 4\,\cancel{n\pi}\,\cancel{(-1)^{2n}}^{\,1} = \boxed{\frac{4}{n^2\pi^2} = a_n}$$

3. Calculate the b_n

(calculation is similar → put in notes)

$$b_n = \int_0^2 t^2 \sin(n\pi t)\,dt$$

$\vdots$

$$\boxed{b_n = \frac{-4}{n\pi}}$$

so we have:

$$a_0 = \frac{8}{3} \qquad a_n = \frac{4}{n^2\pi^2} \qquad b_n = \frac{-4}{n\pi}$$

Putting it all together

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos(n\pi t) + b_n \sin(n\pi t)$$

$$\boxed{f(t) = \frac{4}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2\pi^2}\cos(n\pi t) - \frac{4}{n\pi}\sin(n\pi t)}$$

Ex: Drawing conclusions from the F.S.

let $t=0$

$$f(0) = 2 = \frac{4}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2\pi^2}\cancel{\cos(0)}^{\,1} - \cancel{\frac{4}{n\pi}\sin(0)}^{\,0}$$

$$2 = \frac{4}{3} + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}$$

$$2 - 3\cdot 2 - \frac{4}{3} = \frac{4}{\pi^2}\sum^{\infty}\frac{1}{n^2}$$

$$\frac{2}{3} = \frac{3\cdot 2}{3} - \frac{4}{3} = \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}$$

$$\frac{\pi^2}{6} = \frac{2}{3}\cdot\frac{\pi^2}{4} = \sum_{n=1}^{\infty}\frac{1}{n^2}$$

$$\boxed{\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}}$$

This series was discovered by Euler