Lecture 13

★ Section 9.3

Fourier Sine & Cosine Series

Announcements:
HW + A3 due Tues Jul 6
Goal: finish grading Midterm 1 by next Friday

Warmup:

Recall, a function $f(t)$ is said to be:

<u>even</u> if $f(-t) = f(t)$ for all t

<u>odd</u> if $f(-t) = -f(t)$ for all t

Determine if the following functions are even or odd (or neither)

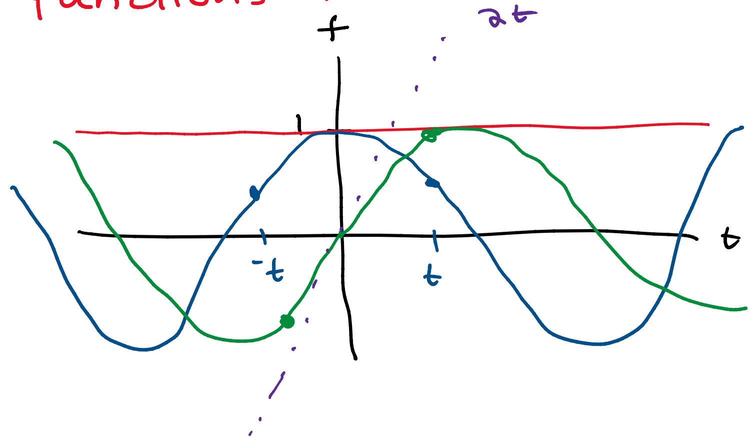
(a) $f(t) = 1$ even

(b) $f(t) = \cos(t)$ even

(c) $f(t) = \sin(t)$ odd

(d) $f(t) = \underbrace{2t}_{\text{odd}} + \underbrace{3\sin(t)}_{\text{odd}}$

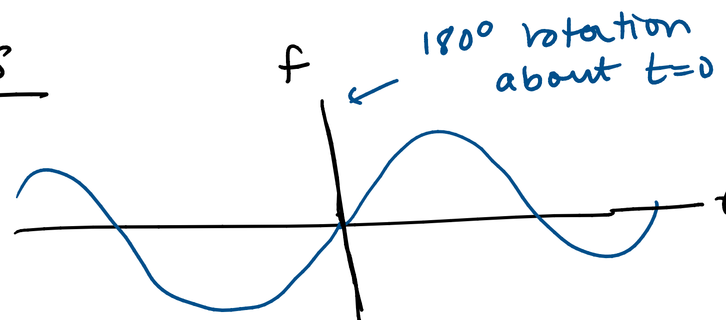
so $f(t)$ is also odd



I. Even and Odd Extensions

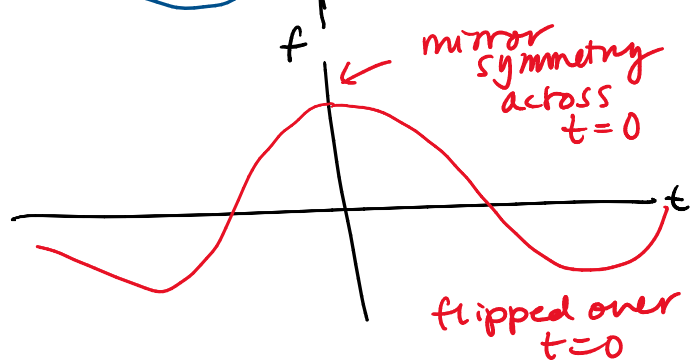
$\sin(t)$ is odd

$\sin(-t) = -\sin(t)$



$\cos(t)$ is even

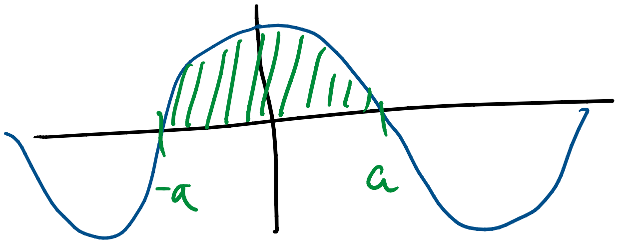
$\cos(-t) = \cos(t)$



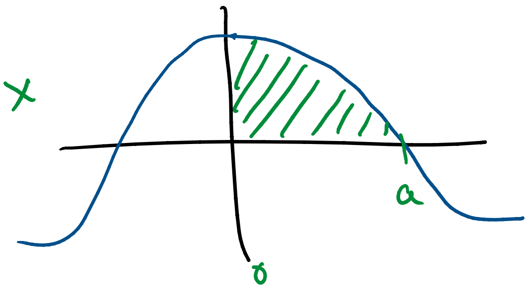
Furthermore, if $f(t)$ is

even: $\int^{a} f(t)\,dt = 2\int_0^a f(t)\,dt$

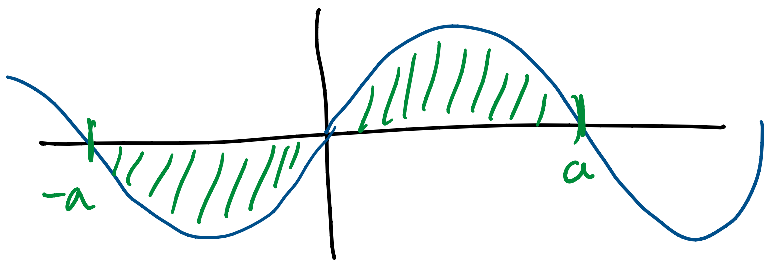
even: $\int_{-a}^{a} f(t)\,dt = 2\int_0 f(t)\,dt$



$= 2\times$



odd: $\int_{-a}^{a} f(t)\,dt = 0$



these two regions cancel out

NOTE: If $f(t)$ is a periodic and even function then its F.S. will have only cosine terms

$$\underbrace{f(t)}_{\text{even}} \simeq \underbrace{\frac{a_0}{2}}_{\text{even}} + \sum_{n=1}^{\infty} \underbrace{a_n \cos\left(\frac{n\pi t}{L}\right)}_{\text{even}} + \cancel{b_n \underbrace{\sin\left(\frac{n\pi t}{L}\right)}_{\text{odd}}}^{\,0}$$

$$\boxed{b_n = 0}$$

Similarly, if $f(t)$ is periodic and odd, its F.S. will have only sine terms

$$\underbrace{f(t)}_{\text{odd}} \simeq \underbrace{\cancel{\frac{a_0}{2}}^{\,0}}_{\text{even}} + \sum_{n=1}^{\infty} \underbrace{\cancel{a_n \cos\left(\frac{n\pi t}{L}\right)}^{\,0}}_{\text{even}} + b_n \underbrace{\sin\left(\frac{n\pi t}{L}\right)}_{\text{odd}}$$

$$\boxed{a_0 = 0} \qquad \boxed{a_n = 0}$$

Consider $f(t)$ defined on one half-period L

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$f(t)$ on $0 < t < L$

GOAL: expand $f(t)$ to be a periodic function for all with $p = 2L$

Two choices:

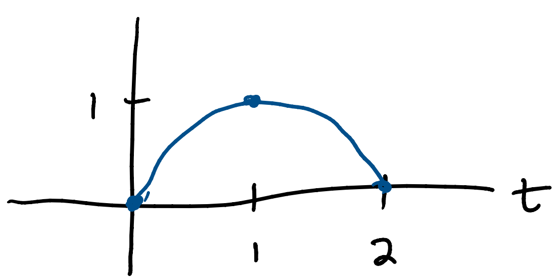
The even period $2L$ extension of f

$$f_E(t) = \begin{cases} f(t) & 0 < t < L \\ f(-t) & -L < t < 0 \end{cases}$$

The odd period $2L$ extension of f

$$f_O(t) = \begin{cases} f(t) & 0 < t < L \\ -f(-t) & -L < t < 0 \end{cases}$$

Ex: $f(t) = 2t - t^2$ on $0 < t < 2$ $\boxed{L = 2}$



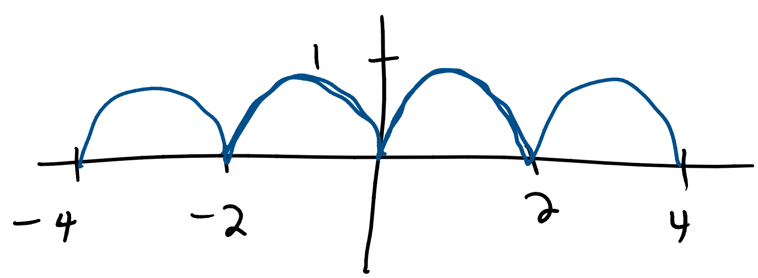
@ $t=0$ $f=0$

@ $t=1$ $f=1$

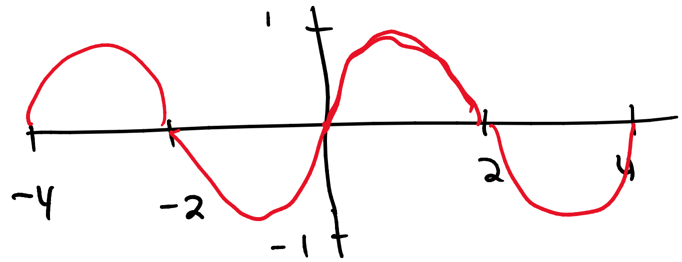
@ $t=2$ $f=0$

quadratic polynomial

The even extension $f_E(t)$



The odd extension $f_o(t)$



Def: Suppose $f(t)$ is piecewise continuous on the interval $[0, L]$. Then the Fourier Cosine Series of f is:

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi t}{L}\right)$$

with $a_n = \frac{2}{L}\int_0^L f(t)\cos\left(\frac{n\pi t}{L}\right)dt$

NOTE: Here, we made the even extension of f

$$f_E(t) = \begin{cases} f(t) & 0 < t < L \\ f(-t) & -L < t < 0 \end{cases}$$

since $f_E(t)$ is even $\longrightarrow$ $b_n = 0$

then $a_n = \frac{1}{L}\int_{-L}^{L} f_E(t)\cos\left(\frac{n\pi t}{L}\right)dt$

because $f_E(t)$ and cos are even, so is $f_E(t)\cos\left(\frac{n\pi t}{L}\right)$ is also even

$$a_n = \frac{1}{L}\int_{-L}^{L} f_E(t)\cos\left(\frac{n\pi t}{L}\right)dt = \frac{2}{L}\int_0^L f_E(t)\cos\left(\frac{n\pi t}{L}\right)dt$$

Similarly, the Fourier Sine Series of f

$$f(t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi t}{L}\right)$$

$$\frac{2}{L}\int_0^L f(t)\sin\left(\frac{n\pi t}{L}\right)dt$$

$$f(t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi t}{L}\right)$$

where $b_n = \frac{2}{L}\int_0^L f(t)\sin\left(\frac{n\pi t}{L}\right) dt$

<u>NOTE:</u> we make the odd extension $f_o(t)$

$\rightarrow$ $a_n = 0$ in F.S.

$$b_n = \frac{1}{L}\int_{-L}^{L} \underbrace{\underbrace{f(t)}_{\text{odd}}\ \underbrace{\sin\left(\frac{n\pi t}{L}\right)}_{\text{odd}}}_{\text{even}} dt = \frac{2}{L}\int_0^L f(t)\sin\left(\frac{n\pi t}{L}\right)dt$$

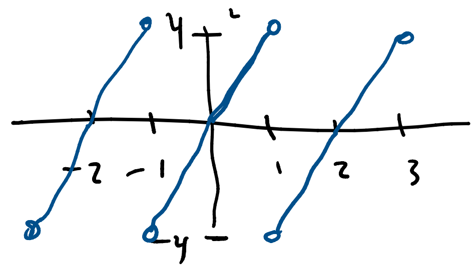
$\boxed{L=1}$

<u>Ex</u>: Let $f(t) = 4t$ for $0<t<1$

Find the <u>Fourier Sine Series</u> of $f(t)$

1. Make the <u>odd</u> extension

$$f_o(t) = \begin{cases} 4t & 0<t<1 \\ 4t & -1<t<0 \end{cases}$$



2. $f_o(t)$ is odd, so we know $a_0 = 0$ and $a_n = 0$

3. Find b_n

$$b_n = \frac{2}{L}\int_0^L f(t)\sin\left(\frac{n\pi t}{L}\right)dt = \frac{2}{1}\int_0^1 (4t)\sin\left(\frac{n\pi t}{1}\right)dt$$

u substitution

$u = n\pi t \qquad t = \frac{u}{n\pi}$

$dt = \frac{du}{n\pi}$

$$= \frac{8}{(n\pi)^2}\int_0^{n\pi} u\sin(u)\,du$$

from Sec 9.1, use

$\int u\sin(u)\,du = -u\cos(u) + \sin(u)$

$$= \frac{8}{(n\pi)^2}\left[-u\cos(u) + \sin(u)\right]\Big|_0^{n\pi}$$

$$= \frac{8}{n^2\pi^2}\left[-u\cos(u)+\sin(u)\right]\Big|_0^{n\pi}$$

$$= \frac{8}{n^2\pi^2}\left[-(n\pi)\underbrace{\cancel{\cos(n\pi)}}_{(-1)^n} + \underbrace{\cancel{\sin(n\pi)}}_{0} - \underbrace{0\cdot\cancel{\cos(0)}}_{0} - \underbrace{\cancel{\sin(0)}}_{0}\right]$$

$$= \frac{-8\cancel{n\pi}(-1)^n}{\cancel{n^2\pi^2}} = \boxed{\frac{8(-1)^{n+1}}{n\pi} = b_n}$$

So the Fourier Sine Series of $f(t)=4t$

$$\boxed{4t = f(t) = \sum_{n=1}^{\infty} \frac{8(-1)^{n+1}}{n\pi}\sin(n\pi t)}$$

II. Solving Differential Equations:

GOAL: Represent solutions of ODEs and PDEs in terms of Fourier Series

Def: An endpoint value problem is an ODE with the following conditions

$$\begin{cases} ax'' + bx' + cx = f(t) \\ x(0) = x(L) = 0 \end{cases} \qquad (0 < t < L)$$

want solutions $x(t)$ on the interval $[0, L]$

NOTE: In contrast, an initial value problem (IVP) has conditions $x(0)=a$, $x'(0)=b$

To solve this using F.S.

1. extend $f(t)$ to be periodic $(P=2L)$

1. extend $f(t)$ to be periodic $(p = 2L)$

2. Find the F.S. of $f(t)$

$$f(t) = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi t}{L}\right) + B_n \sin\left(\frac{n\pi t}{L}\right)$$

Fourier coefficients are known

3. Assume $x(t)$ also has a F.S.

$$x(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi t}{L}\right) + b_n \sin\left(\frac{n\pi t}{L}\right)$$

Fourier Coeff are <u>unknown</u>

<u>GOAL</u>: Find a_n and b_n

4. plug $x(t)$ into ODE and solve for a_n and b_n.

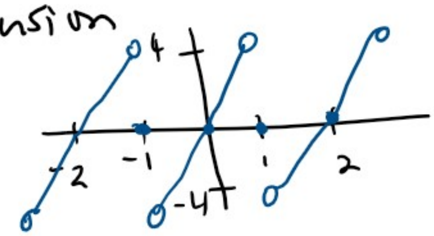
<u>EX</u>: Find a Fourier Series Solution to the endpoint value problem:

$$\begin{cases} x'' + 4x = 4t \\ x(0) = x(1) = 0 \end{cases} \qquad 0 < t < 1$$

Here $f(t) = 4t$ on $0 < t < 1$

1. Extend $f(t)$ to be the <u>odd</u> extension

$$f_o(t) = \begin{cases} 4t & 0 < t < 1 \\ 4t & -1 < t < 0 \end{cases}$$



<u>NOTE</u>: Which extension should we choose

<u>odd</u> if endpoint conditions $x(0) = x(L) = 0$

conditions

even if end point conditions $x'(0) = x'(L) = 0$

2. Find the F.S. of $f_o(t)$

found this earlier $b_n = \frac{8(-1)^{n+1}}{n\pi}$

$$4t = \sum_{n=1}^{\infty} \frac{8(-1)^{n+1}}{n\pi} \sin(n\pi t)$$

Fourier Sine Series

3. Assume $x(t)$ has a Fourier Sine Series

$$x(t) = \sum_{n=1}^{\infty} b_n \sin(n\pi t)$$

GOAL: Find b_n

4. Plug $x(t)$ into the ODE

$$x'' + 4x = 4t$$

$$\frac{d^2}{dt^2}\left(\sum_{n=1}^{\infty} b_n \sin(n\pi t)\right) + 4\left(\sum_{n=1}^{\infty} b_n \sin(n\pi t)\right) = \sum_{n=1}^{\infty} \frac{8(-1)^{n+1}}{n\pi} \sin(n\pi t)$$

take derivatives termwise

$$\left(\sum_{n=1}^{\infty} b_n(-n^2\pi^2)\sin(n\pi t)\right) + \left(\sum_{n=1}^{\infty} 4b_n \sin(n\pi t)\right) = \sum_{n=1}^{\infty} \frac{8(-1)^{n+1}}{n\pi} \sin(n\pi t)$$

all of these terms are linear
→ combine like terms into one sum

$$\sum_{n=1}^{\infty} \left[(-n^2\pi^2 + 4)b_n - \frac{8(-1)^{n+1}}{n\pi}\right] \sin(n\pi t) = 0$$

in order for

$\sum_{n=1} [\ldots n\pi]$

these terms must sum to zero for all n

in order for this to be zero

$$(-n^2\pi^2+4)b_n - \frac{8(-1)^{n+1}}{n\pi} = 0 \quad \text{for all } n$$

solve for b_n

$$\boxed{b_n = \frac{8(-1)^{n+1}}{n\pi(4-n^2\pi^2)}}$$

So now, we can write the solution $x(t)$ to the ODE

$$\boxed{x(t) = \sum_{n=1}^{\infty} \frac{8(-1)^{n+1}}{n\pi(4-n^2\pi^2)} \sin(n\pi t)}$$

★ Summarize:

We can use Fourier Series to solve ODEs of the form:

$$\begin{cases} ax'' + bx' + cx = f(t) & 0 < t < L \\ x(0) = x(L) = 0 \end{cases}$$

→ solution $x(t)$ is a Fourier Sine Series

make odd extension $f_o(t)$

$$\begin{cases} ax'' + bx' + cx = f(t) \\ x'(0) = x'(L) = 0 \end{cases}$$

→ solution $x(t)$ is Fourier Cosine Series

[illegible] $f_e(t)$

$\rightarrow$ solution $x(t)$ is Fourier cosine series

make <u>even</u> extension $f_E(t)$