Lecture 14

☆ Section 9.4 - Part 1

Applications of Fourier Series

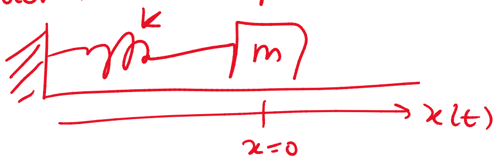
Announcements

Online HW (9.1 + 9.2) and A3 due Today @ 11:59 pm

Office Hours Today @ 2:30-3:30pm

Warm up:

Consider the undamped mass-on-a-spring system



mass m

spring constant k

displacement $x(t)$

equation: $mx'' + kx = 0$

What is the natural frequency ω_0?

Ans: $\omega_0 = \sqrt{\frac{k}{m}}$

comes from solving the ODE

char eq: $mr^2 + k = 0$

roots: $r^2 = -\frac{k}{m}$ $\quad r = \pm i\sqrt{\frac{k}{m}} = \pm i\omega_0$

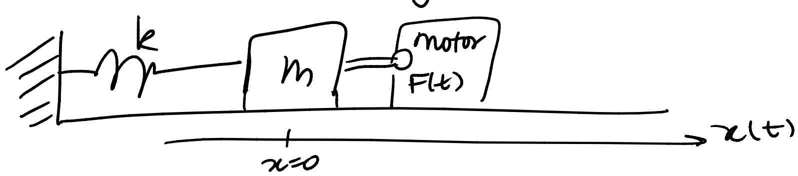
general solution

$x_c(t) = C_1\cos(\omega_0 t) + C_2\sin(\omega_0 t)$

I. Forced Oscillations:

Undamped mass-on-a-spring

Now add a forcing function $F(t)$



equation: $mx'' + kx = F(t)$

From MA 266, we know that we can write the solution as:

$$x(t) = \underbrace{x_c(t)}_{\text{complementary solution}} + \underbrace{x_{sp}(t)}_{\text{steady periodic solution}}$$

Complementary solution: solves homogeneous ODE ($F(t)=0$)

steady periodic solution: solves the non homogeneous ODE

$$x(t) = C_1\cos(\omega_0 t) + C_2\sin(\omega_0 t) + x_{sp}(t)$$

If $F(t) = F_0 \cos(\omega t)$ — cosine forcing term

solve for $x_{sp}(t)$ by the Method of Undetermined Coefficients

Ansatz: $x_{sp}(t) = A\cos(\omega t) + B\sin(\omega t)$

A is an unknown constant

Plug $x_{sp}(t)$ into the ODE and solve for A.

(In this B happens to be B=0)

$$x_{sp}' = -\omega A \sin(\omega t)$$

$$x_{sp}'' = -\omega^2 A \cos(\omega t)$$

$$mx'' + kx = F_0 \cos(\omega t)$$

$$m\left[-\omega^2 A \cos(\omega t)\right] + k\left[A\cos(\omega t)\right] = F_0\cos(\omega t)$$

$$\underbrace{\left[-m\omega^2 A + kA - F_0\right]}_{\text{must sum to zero}} \cos(\omega t) = 0$$

$$\left(k - m\omega^2\right]A = F_0 \qquad A = \frac{F_0}{k - m\omega^2}$$

$$\boxed{x_{sp}(t) = \frac{F_0}{k - m\omega^2}\cos(\omega t)}$$

called the steady periodic solution b/c it oscillates for all t,

Full solution

$$x(t) = C_1\cos(\omega_0 t) + C_2\sin(\omega_0 t) + \frac{F_0}{k - m\omega^2}\cos(\omega_0 t)$$

Q: What if the forcing term $F(t)$ is periodic but not a sine or cosine?

A: Represent $F(t)$ as a Fourier Series

Assume $F(t)$ is an odd periodic function $P = 2L$

$$F(t) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi t}{L}\right)$$

(*) Under most conditions, we can a solution $x_{sp}(t)$

(*) under **most** conditions, we can a solution $x_{sp}(t)$

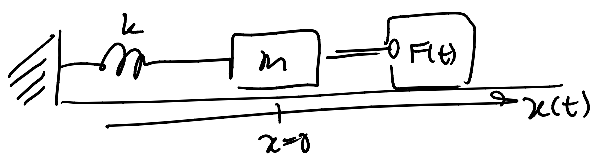
Idea: for each term in the series, solve the corresponding ODE using the method of undetermined coefficients

Ansatz: $x_{sp}(t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi t}{L}\right)$

Here, the b_n are undetermined (unknown)

Plug into ODE and solve for b_n

Ex:



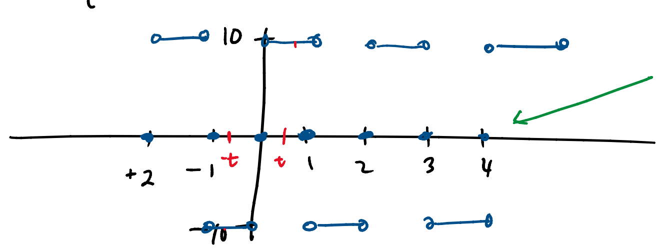
$m = 2\,\text{kg}$, $k = 32\ \text{N/m}$

F(t) is odd periodic function $P = 2$ $L = 1$

$$F(t) = \begin{cases} +10\,\text{N} & \text{if } 0 < t < 1 \\ -10\,\text{N} & \text{if } 1 < t < 2 \end{cases}$$

$f(-t) = -f(t)$
odd function

plot:



F(t) = 0 at points of discontinuity so that the F.S. converges

Then solve: $2x'' + 32x = F(t)$

the natural frequency $\omega_0 = \sqrt{\frac{k}{m}} = \sqrt{\frac{32}{2}} = 4$

complementary solution $x_c(t) = C_1 \cos(4t) + C_2 \sin(4t)$

GOAL: Find $x_{sp}(t)$

1. Find the F.S. of $F(t)$
   since F(t) is odd $\longrightarrow$ Fourier Sine Series
   $A_0 = 0$ $A_n = 0$ $L = 1$
   $\ldots = 2\int_0^1 10 \sin(n\pi t)\,dt$

since F(t) is

$A_0 = 0 \qquad A_n = 0 \qquad L = 1$

$$B_n = \frac{2}{L}\int_0^L f(t)\sin\left(\frac{n\pi t}{L}\right)dt = 2\int_0^1 10\sin(n\pi t)\,dt$$

$$\ldots = \begin{cases} \frac{40}{n\pi} & \text{if } n \text{ odd} \\ 0 & \text{if } n \text{ even} \end{cases}$$

$$F(t) = \sum_{n\ odd} \frac{40}{n\pi}\sin(n\pi t)$$

2. Assume $x_{sp}(t) = \sum_{n\ odd} b_n \sin(n\pi t)$

only expect $x_{sp}(t)$ to have the same n terms as $F(t)$

3. Plug $x_{sp}(t)$ into the ODE

$$2x'' + 32x = F(t)$$

$$2\frac{d^2}{dt^2}\left(\sum_{n\ odd} b_n \sin(n\pi t)\right) + 32\left(\sum_{n\ odd} b_n\sin(n\pi t)\right) = \left(\sum_{n\ odd}\frac{40}{n\pi}\sin(n\pi t)\right)$$

distribute

$$\left(\sum_{n\ odd} 2b_n(-n^2\pi^2)\sin(n\pi t)\right) + \left(\sum_{n\ odd} 32b_n\sin(n\pi t)\right) - \left(\sum_{n\ odd}\frac{40}{n\pi}\sin(n\pi t)\right)$$

collect like terms into one sum

$$\sum_{n\ odd}\underbrace{\left[-2n^2\pi^2 b_n + 32b_n - \frac{40}{n\pi}\right]}_{\text{each of these terms must equal zero}}\sin(n\pi t) = 0$$

$$(32 - 2n^2\pi^2)b_n - \frac{40}{n\pi} = 0 \quad \text{for each } n$$

$$(32 - 2n^2\pi^2)b_n = \frac{40}{n\pi}$$

$$b_n = \frac{40}{n\pi(32 - 2n^2\pi^2)} = \boxed{\frac{20}{n\pi(16 - n^2\pi^2)} = b_n}$$

Steady periodic solution:

$$\boxed{x_{sp}(t) = \sum_{n\ odd}\frac{20}{n\pi(16 - n^2\pi^2)}\sin(n\pi t)}$$

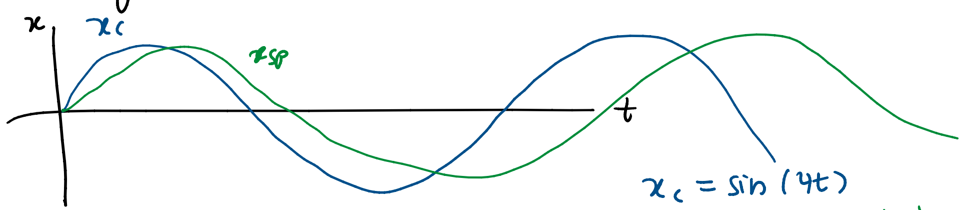
And the full solution is

$$x(t) = x_c(t) + x_{sp}(t)$$

$$x(t) = C_1\cos(4t) + C_2\sin(4t) + \sum_{n\ odd}\frac{20\sin(n\pi t)}{n\pi(16 - n^2\pi^2)}$$

$$x(t) = \underbrace{C_1\cos(4t) + C_2\sin(4t)}_{\substack{x_c(t) \text{ has a period} \\ 4p = 2\pi \\ p = \frac{2\pi}{4} = \frac{\pi}{2} \sim 1.6}} + \underbrace{\sum_{n \text{ odd}} \frac{20\sin(n\pi t)}{n\pi(16 - n^2\pi^2)}}_{\substack{\text{has a period } p=2 \\ (\text{same as } F(t))}}$$

The frequencies will interact causing "beats"



$x_c = \sin(4t)$

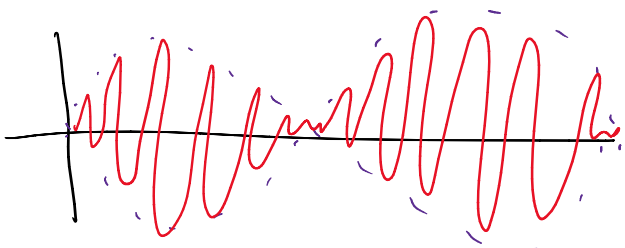
$x_{sp} = \frac{20}{\pi(16-\pi^2)}\sin(\pi t)$

($n=1$ term)

Add $x_c(t) + x_{sp}(t)$

Beats explained on page 199 of textbook

Also see top of page 594



$2\sin(\theta)\sin(\varphi) = \sin(\theta+\varphi) + \sin(\theta-\varphi)$

$x = x_c + x_{sp} = \sin(4t) + \frac{20}{\pi(16-\pi^2)}\sin(\pi t)$

using trig formula

$= F\underbrace{\sin((4+\pi)t)}_{\text{slow oscillation}}\underbrace{\sin((4-\pi)t)}_{\text{fast oscillation}}$

$\boxed{\text{Sec 3.6 in Text book}}$

fast oscillation

amplitude oscillates with a slower frequency

II. Pure Resonance:

Recall, with a cosine forcing term

$$mx'' + kx = F_0\cos(\omega_0 t)$$

forcing frequencies equal to natural frequency

$$\omega = \omega_0 = \sqrt{\frac{k}{m}}$$

we get pure resonance

If we try to solve using Undetermined Coefficients

Ansatz: $x_{sp}(t) = A\cos(\omega_0 t)$

if we plug into ODE, problem

$x_c(t) = C_1\cos(\omega_0 t) + C_2\sin(\omega_0 t)$

$$m x_{sp}'' + k x_{sp} = 0 \neq F(t) = F_0\cos(\omega_0 t)$$

$$0 = F_0 \quad \Rightarrow\Leftarrow$$

Need to Adjust our Ansatz: (Chap 3.5) Nonhomogeneous Eqns + Undetermined Coefficients

Rule of Thumb: $x_{sp}(t) = A\,t\cos(\omega_0 t)$

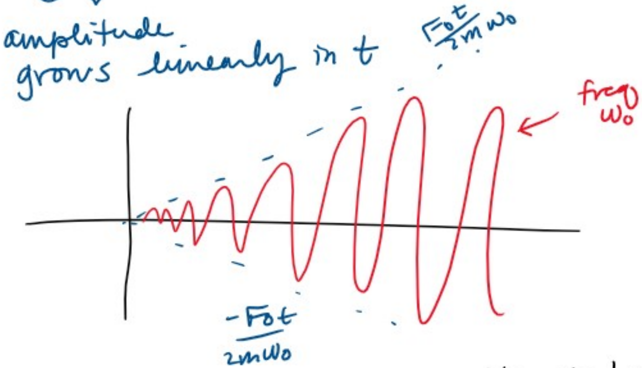
multiply by a factor of t

plug into ODE & solve for A

$$\boxed{x_{sp}(t) = \underbrace{\frac{F_0}{2m\omega_0}}_{\text{amplitude grows linearly in } t}\, t\sin(\omega_0 t)}$$

when $\omega = \omega_0$

← see example on page 200 in textbook



amplitude of oscillation grows without bound

→ pure resonance

★ The same thing can happen if the $F(t)$ is a Fourier Series

let $F(t) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi t}{L}\right)$

frequency $\omega_n = \frac{n\pi}{L}$

ODE: $mx'' + kx = F(t)$ $\qquad \omega_0 = \sqrt{\frac{k}{m}}$

If there is an index N such that the frequency of F.S. terms is:

$$\omega_N = \frac{N\pi}{L} = \omega_0$$

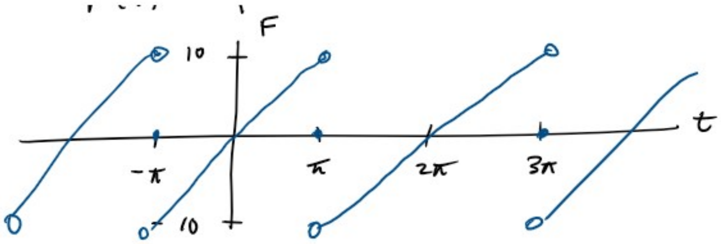
then this term causes pure resonance

Ex: $2x'' + 32x = F(t)$ $\qquad \omega_0 = \sqrt{\frac{32}{2}} = 4$

$F(t) = \{\, 10t$ on $-\pi < t < \pi$

$P = 2\pi$

$L = \pi$



$P = 2\pi$
$L = \pi$
odd function
Fourier Sine Series

Here $F(t) = \sum_{n=1}^{\infty} \frac{20(-1)^{n+1}}{n} \sin(nt)$

$\omega_n = n$ for $n = 1, 2, 3, \dots$

Q: Does this system have pure resonance?

is there an N s.t. $N = \omega_N \stackrel{?}{=} \omega_0 = 4$

$$\boxed{N = 4}$$

yes, pure resonance when $N = 4$

$$x_{sp}(t) = \underbrace{\frac{5t}{8}\cos(4t)}_{\text{pure resonance term}} + \sum_{n \neq 4} \frac{10(-1)^{n+1}}{n(16-n^2)} \sin(nt)$$

this example is in Textbook in Sec 9.4 page 594

Solve:

$2x'' + 32x = B_4 \sin(4t)$

$x = A_4 t \sin(4t)$

plug into ODE and solve for A_4