Lecture 15

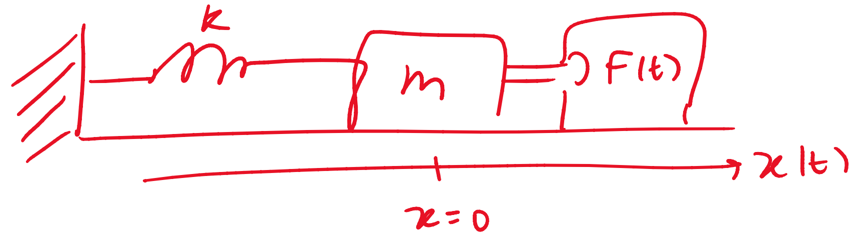
Section 9.4 - Part 2

Applications of Fourier Series

Announcements:
Online HW + A4 due Tues Jul 13
Office Hours Today @ 2:30-3:30pm

Warm up:
mass-on-a-spring system

$$mx'' + kx = F(t) = F_0 \cos(\omega t)$$

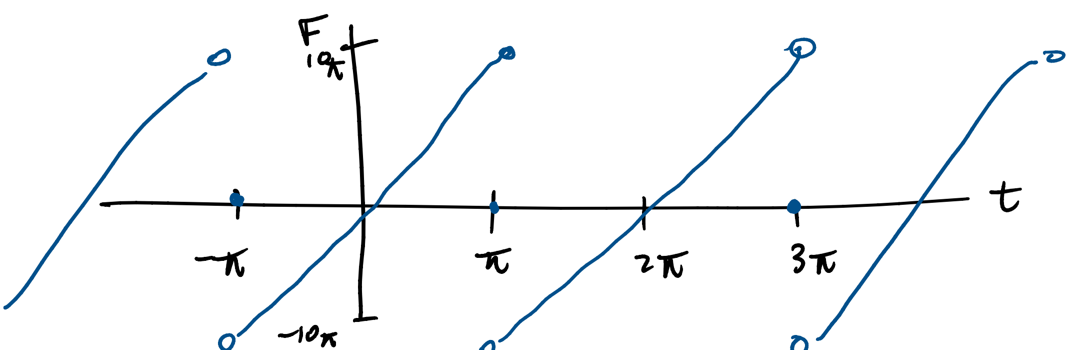


Recall, pure resonance occurs when the external forcing frequency ω matches the natural frequency $\omega_0 = \sqrt{\frac{k}{m}}$

I. Pure Resonance:

$m=2 \quad k=32 \longrightarrow \omega_0 = \sqrt{\frac{k}{m}} = \sqrt{\frac{16}{2}} = 4$

$F(t) = \{ 10t$ on $-\pi < t < \pi \}$ 2π periodic function



Here $F(t)$ is odd & periodic $\longrightarrow$ Fourier Sine Series

$$F(t) = \sum_{n=1}^{\infty} \frac{20(-1)^{n+1}}{n} \sin(nt)$$

external frequency $\omega_n = n$

$\omega_4 = \omega_0$
$\rightarrow$ pure resonance

Solve ODE: $2x'' + 32x = F(t)$

b_n unknown

Solve ODE: $2x'' + 32x = F(t)$ → pure resonance

Assume $x_{sp}(t) = \sum_{n=1}^{\infty} b_n \sin(nt)$ b_n unknown WANT to find

plug into ODE and solve for b_n

$$2\frac{d^2}{dt^2}\left(\sum_{n=1}^{\infty} b_n \sin(nt)\right) + 32\left(\sum_{n=1}^{\infty} b_n \sin(nt)\right) = \left(\sum_{n=1}^{\infty} \frac{20(-1)^{n+1}}{n}\sin(nt)\right)$$

$$\left(\sum_{n=1}^{\infty} 2b_n(-n^2)\sin(nt)\right) + \left(\sum_{n=1}^{\infty} 32 b_n \sin(nt)\right) = \left(\sum_{n=1}^{\infty} \frac{20(-1)^{n+1}}{n}\sin(nt)\right)$$

Combine like terms

$$\sum_{n=1}^{\infty} \underbrace{\left[-2n^2 b_n + 32 b_n - \frac{20(-1)^{n+1}}{n}\right]}_{\text{must equal zero}} \sin(nt) = 0$$

$$-2n^2 b_n + 32 b_n - \frac{20(-1)^{n+1}}{n} = 0 \quad \text{for all } n$$

$$(32 - 2n^2) b_n = \frac{20(-1)^{n+1}}{n}$$

$$b_n = \frac{20(-1)^{n+1}}{n(32-2n^2)} = \boxed{\frac{10(-1)^{n+1}}{n(16-n^2)} = b_n}$$

NOTE: Problem! When $n=4$

$$b_4 = \frac{10(-1)^{4+1}}{4(16-4^2)} = \frac{10(-1)^5}{0}$$ division by zero → undefined

This is because when $n=4$ the forcing term $F(t)$ has $\sin(4t)$

$\omega_4 = 4$

the forcing term F(t) has ...

$$\omega_4 = 4$$

But, the natural frequency $\omega_0 = \sqrt{\frac{k}{m}} = 4$

complementary soln

$$x_c(t) = C_1 \cos(4t) + \underbrace{C_2 \sin(4t)}$$

$\sin(4t)$ solves the homogeneous eqn

can't also solve the non-homog. eqn

$$\text{So } x_{sp}(t) = \sum_{n \neq 4} \frac{10(-1)^{n+1} \sin(nt)}{n(16-n^2)} + \overbrace{x_4(t)}^{\text{new ansatz}}$$

New Ansatz: $x_4(t) = At\cos(4t) + Bt\sin(4t)$

(this is covered in detail in Sec 3.5)

plug $x_4(t)$ into the ODE:

$$2x_4'' + 32x_4 = B_4 \sin(4t) = \frac{20(-1)^{4+1}}{4} \sin(4t)$$

$$x_4'' = (8B - 16At)\cos(4t) + (-8A - 16Bt)\sin(4t)$$

$$2\left[(8B - 16At)\underline{\cos(4t)} + (-8A - 16Bt)\underline{\sin(4t)}\right]$$

$$+ 32\left[At\underline{\cos(4t)} + Bt\underline{\sin(4t)}\right] = -5\underline{\sin(4t)}$$

collect like terms

$$\left[16B - 32\cancel{At} + 32\cancel{At}\right]\cos(4t)$$

$$+ \left[-16A - 32\cancel{Bt} + 32\cancel{Bt} + 5\right]\sin(4t) = 0$$

$$-16A + 5 = 0$$

$16B = 0$

$B = 0$

$-16A + 5 = 0$

$A = \frac{5}{16}$

$$x_4(t) = \underbrace{\frac{5}{16}t}_{\text{amplitude grows w/o bound}} \underbrace{\cos(4t)}_{\text{oscillation w/ freq } w=4} \leftarrow \text{pure resonance}$$

so the steady periodic solution

$$\boxed{x_{sp}(t) = \frac{5t}{16}\cos(4t) + \sum_{n \neq 4} \frac{10(-1)^{n+1}}{n(16-n^2)} \sin(nt)}$$

I. Damped Forced Oscillations

mass - spring - dashpot system

c - damping

$$mx'' + cx' + kx = F(t)$$

Again solution has form

$$x(t) = \underbrace{x_c(t)}_{\text{solves homog eq}} + \underbrace{x_{sp}(t)}_{\text{solves the non-homogeneous ODE}}$$

- underdamped
- critically damped
- overdamped

oscillate for all t

$$x_c(t) \xrightarrow[t \to \infty]{} 0$$

If $F(t)$ is odd and periodic with $P = 2L$

Fourier Sine Series

If $F(t)$ is odd and periodic ...

it has a Fourier sine series

$$F(t) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi t}{L}\right)$$

Want to solve

$$m x'' + c x' + k x = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi t}{L}\right)$$

$$\omega_n = \frac{n\pi}{L}$$

For each n, solve

$$m x_n'' + c x_n' + k x_n = B_n \sin(\omega_n t)$$

where $\omega_n = \frac{n\pi}{L}$ is the forcing freq

$$x_{sp}(t) = \sum_{n=1}^{\infty} x_n$$

From Sec 3.6, we know $x_n(t)$ looks like

Derived in Sec 3.6

$$x_n(t) = \frac{B_n \sin(\omega_n t - \alpha_n)}{\sqrt{(k - m\omega_n^2)^2 + (c\omega_n)^2}}$$

where $\omega_n = \frac{n\pi}{L}$ $\qquad \alpha_n = \tan^{-1}\left(\frac{c\omega_n}{k - m\omega_n^2}\right)$

$$\boxed{\begin{gathered} x_{sp}(t) = \sum_{n=1}^{\infty} \frac{B_n \sin(\omega_n t - \alpha_n)}{\sqrt{(k - m\omega_n^2)^2 + (c\omega_n)^2}} \\ \omega_n = \frac{n\pi}{L} \qquad \alpha_n = \tan^{-1}\left(\frac{c\omega_n}{k - m\omega_n^2}\right) \end{gathered}}$$

Ex: $m = 3\text{kg}$ $\qquad c = 0.02\ \text{N/m/s}$ $\qquad k = 27\ \text{N/m}$

$$3x'' + 0.02x' + 27x = F(t)$$

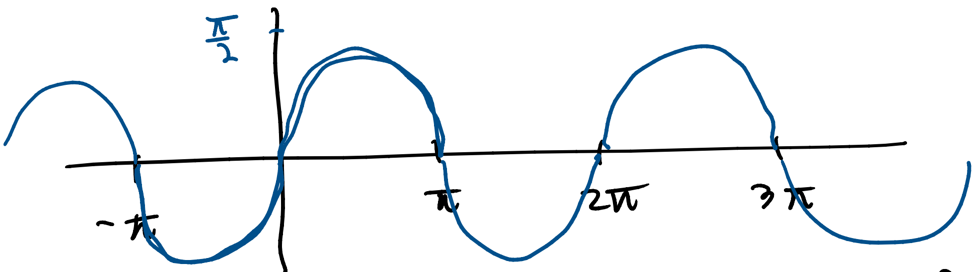
Ex: $m = 3\,kg$

$$3x'' + 0.02x' + 27x = F(t)$$

where $F(t)$ is an odd periodic function defined on $0 < t < L$

$$F(t) = \{ \pi t - t^2 \quad \text{if} \quad 0 < t < \pi \} \qquad L = \pi$$

1. Make the odd extension



Find the Fourier Sine Series of $F(t)$

$$F(t) = \sum_{n \text{ odd}} \frac{8}{\pi n^3} \sin(nt)$$

2. Find the steady periodic solution $x_{sp}(t)$

$m = 3 \qquad c = 0.02 \qquad k = 27 \qquad L = \pi$

$$\omega_n = \frac{n\pi}{L} = \frac{n\pi}{\pi} = n \qquad B_n = \begin{cases} \frac{8}{\pi n^3} & n \text{ odd} \\ 0 & n \text{ even} \end{cases}$$

$$\boxed{\alpha_n = \tan^{-1}\left(\frac{c\omega_n}{k - m\omega_n^2}\right) = \tan^{-1}\left(\frac{0.02n}{27 - 3n^2}\right)}$$

$$x_{sp}(t) = \sum_{n=1}^{\infty} \frac{B_n \sin(\omega_n t - \alpha_n)}{\sqrt{(k - m\omega_n^2)^2 + (c\omega_n)^2}}$$

$$\boxed{x_{sp}(t) = \sum_{n \text{ odd}} \frac{\left(\frac{8}{\pi n^3}\right)\sin(nt - \alpha_n)}{\sqrt{(27-3n^2)^2 + (0.02n)^2}}}$$

Using a calculator, let's find the first few terms in series

$\boxed{n=1}$

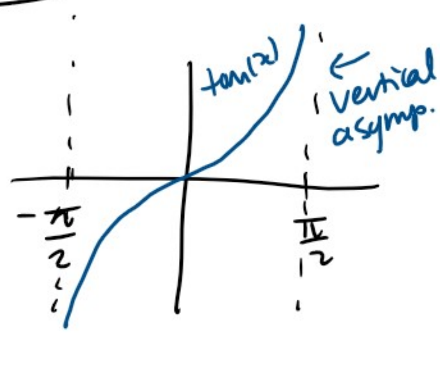
$$\alpha_1 = \tan^{-1}\left(\frac{0.02(1)}{27-3(1)^2}\right) = \tan^{-1}\left(\frac{0.02}{24}\right) \approx 0.0008$$

$$b_1 = \frac{\left(\frac{8}{\pi(1)^3}\right)}{\sqrt{(27-3(1)^2)^2 + (0.02(1))^2}} \approx 0.1061$$

$$x_{sp}(t) \sim 0.1061 \sin(t - 0.0008) + \dots$$

$\boxed{n=3}$

$$\alpha_3 = \tan^{-1}\left(\frac{0.02(3)}{27-3(3)^2}\right) = \frac{\pi}{2}$$



$$b_3 = \frac{\left(\frac{8}{\pi(3)^3}\right)}{\sqrt{\cancel{(27-3(3)^2)^2} + ((0.02)(3))^2}}$$

$$= \frac{8}{27\pi(0.02)(3)} \approx 1.5719$$

$$x_{sp}(t) \sim (0.1061)\sin(t - 0.0008)$$
$$+ (1.5719)\sin\left(3t - \frac{\pi}{2}\right)$$
$$\dots 3.1437) + \dots$$

$$+ \boxed{(1.5719)} \sin\left(3t - \tfrac{\pi}{2}\right)$$

$$+ (0.0004) \sin(4t - 3.1437) + \cdots$$

amplitude when $n=3$ is much larger than other terms

this term "dominates" the series

Q: Why is this term so large?

Let's look at the natural freq ω_0 of the <u>undamped</u> system

$$mx'' + kx = 0 \qquad \omega_0 = \sqrt{\frac{k}{m}} = \sqrt{\frac{27}{3}} = 3$$

In the damped case

$$mx'' + cx' + kx = \sum_{n=1}^{\infty} B_n \sin(\omega_n t)$$

$$\omega_3 = 3 = \omega_0$$

if $c = 0 \longrightarrow$ pure resonance

but because $c \neq 0$, the amplitude b_n is large, but does not grow unbounded