Lecture 16

Section 9.5 – Part 1

Heat Conduction & Separation of Variables

Announcements:

Online HW + A4 due Tues July 13

Warm up:

Recall, a partial Differential Equation (PDE) relates a function of multiple variables with its partial derivatives.

Ex: $\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x}$ Here $u(x,t)$ (can also write: $u_t = u_x$)

Which of the following are PDEs?

(a) $u_t = u_{xx}$ ✓

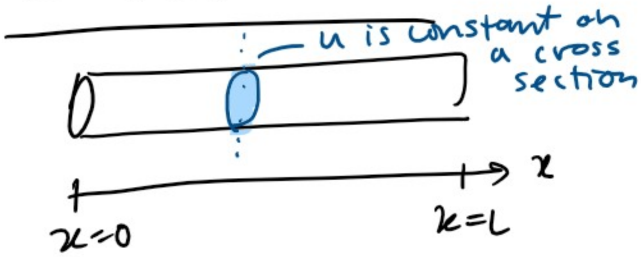
(b) $u_t = u_{xx} + u_{yy}$ ✓

(c) $\frac{du}{dt} = 3 + u$ ✗

$u = u(t)$
ODE

GOAL: Use Fourier Series to solve a PDE

I. Heated Rod:



Rod of length L

$u(x,t)$ – temperature of rod

t – time

Assume that the temperature is constant through a cross section

Model the temperature of rod using the **1D Heat Equation**

$$u_t = k\, u_{xx} \quad \text{on} \quad 0 \le x \le L, \quad t > 0$$

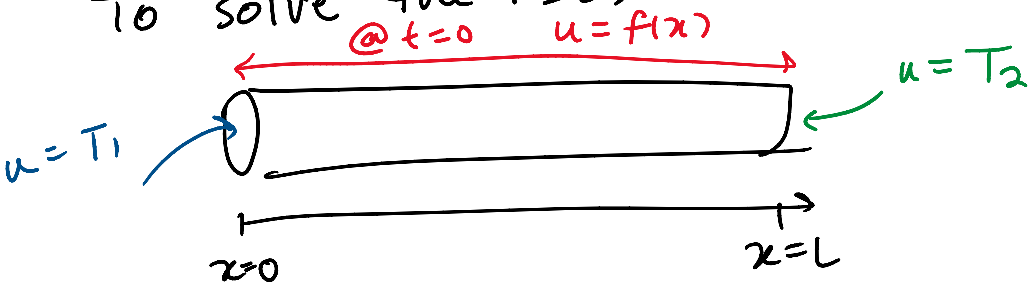
thermal diffusivity $(k > 0)$

Here k is called the <u>thermal</u> <u>diffusivity</u> $(k>0)$
depends on the material of the rod

Common thermal diffusivities (page 604)

material	k (cm^2/s)
Silver	1.70
Copper	1.15
Aluminum	0.85
Iron	0.15
Concrete	0.005

To solve the PDE, need more information



Assume at $x=0$, held at constant temp T_1
at $x=L$, " " " " T_2

At $t=0$, initially the rod has initial temp $f(x)$

<u>Boundary Conditions</u> (BC):

$u(0,t) = T_1$ left side fixed at T_1

Boundary Conditions

$$\begin{cases} u(0,t) = T_1 \\ u(L,t) = T_2 \\ u(x,0) = f(x) \end{cases}$$

left side fixed at T_1

right side fixed at T_2

initial temp @ $t=0$

Boundary Value Problem: (BVP)

$$(*)\quad \begin{cases} u_t = k\,u_{xx} \\ u(0,t) = T_1 \\ u(L,t) = T_2 \\ u(x,0) = f(x) \end{cases} \quad \text{on } 0 \le x \le L,\ t > 0$$

find $u(x,t)$ that satisfies all of these equations

Def If $u(0,t) = u(L,t) = 0$, we say that the PDE has homogeneous boundary conditions

NOTE: The heat eqn $u_t = k\,u_{xx}$ is linear, so the principle of superposition still holds

if $u_1, u_2, u_3, \ldots$ solve $(*)$ with homogeneous BC, then

$$u(x,t) = \sum_{n=1}^{\infty} c_n u_n(x,t)$$ also solves $(*)$

provided:

1. the series converges
2. the series satisfies the initial condition $u(x,0) = f(x)$
3. $u(x,t)$ is continuous

II. Method of Separation of Variables

... used repeatedly in

II. Method of Separation of Variables

NOTE: We will use this method repeatedly in
See 9.5, 9.6, 9.7

GOAL: Solve:
$$\begin{cases} u_t = k\, u_{xx} & \text{for } 0 \le x \le L,\ t > 0 \\ u(0,t) = u(L,t) = 0 & \text{homog. BC.} \\ u(x,0) = f(x) \end{cases}$$

Key step: Assume that $u(x,t)$ can be separated into 2 parts

$$u(x,t) = \underbrace{X(x)}_{\substack{\text{function of} \\ \text{one variable} \\ x}} \underbrace{T(t)}_{\substack{\text{function} \\ \text{of one variable} \\ t}}$$

Intuition: what happens in x doesn't directly affect what happens in time

Plug into PDE: $u_t = k\, u_{xx}$

$$\frac{\partial}{\partial t}\left(X(x)T(t)\right) = k \frac{\partial^2}{\partial x^2}\left(X(x)T(t)\right)$$

pull out (no t dependence)

pull out (b/c no x dependence)

$$X\left(\frac{\partial T}{\partial t}\right) = k\, T\left(\frac{\partial^2 X}{\partial x^2}\right)$$

often, write this as:

$$X T' = k X'' T$$

put all the X terms on one side, and T terms on other

$$\frac{T'}{} = \frac{X''}{}$$

key observation

$$\underbrace{\frac{T'}{kT}}_{\substack{\text{function of } t \\ g(t)}} = \underbrace{\frac{X''}{X}}_{\substack{\text{function of } x \\ h(x)}}$$

key observation

these can only be equal if both sides are a constant

$$g(t) = h(x) = \text{constant} = -\lambda$$

call the constant $-\lambda$ (see why later)

Here λ is called the <u>separation constant</u>

(Assume $\lambda > 0$)

$$\frac{X''}{X} = \frac{T'}{kT} = -\lambda$$

separate into 2 ODEs

$$\frac{X''}{X} = -\lambda$$

$$X'' = -\lambda X$$

$$\boxed{X'' + \lambda X = 0}$$

$$\frac{T'}{kT} = -\lambda$$

$$T' = -k\lambda T$$

$$\boxed{T' + k\lambda T = 0}$$

These are both linear ODEs → solve by finding the characteristic equation in r

Assume $X = e^{rx}$

char eqn

$$r^2 + \lambda = 0$$

roots: $r = \pm i\sqrt{\lambda}$

general solution

$$X = C_1 \cos(\sqrt{\lambda}\, x) + C_2 \sin(\sqrt{\lambda}\, x)$$

Assume $T = e^{rt}$

char eqn:

$$r + k\lambda = 0$$

root: $r = -k\lambda$

general solution

$$T = C_3 e^{-k\lambda t}$$

We also have BC:
$u(0,t) = u(L,t) = 0$

What does this mean for X and T?

$u(0,t) = X(0)T(t) = 0 \Rightarrow X(0) = 0$

$u(L,t) = X(L)T(t) = 0 \Rightarrow X(L) = 0$

endpoint conditions for $X(x)$

$$\begin{cases} X'' + \lambda X = 0 \\ X(0) = 0 \quad X(L) = 0 \end{cases}$$

general soln:

$X = \cancel{C_1 \cos(\sqrt{\lambda} x)}^{\to 0} + C_2 \sin(\sqrt{\lambda} x)$

Plug in BC

$X(0) = 0 = C_1 \cos(0) + C_2 \cancel{\sin(0)}^{0} \longrightarrow C_1 = 0$

$X(L) = 0 = C_2 \sin(\sqrt{\lambda} L)$

Recall $\sin\theta = 0$ if $\theta = n\pi$, $n = 1, 2, 3..$

if $\sqrt{\lambda} L = n\pi$, for $n = 1, 2, 3, \dots$

$$\boxed{\lambda_n = \frac{n^2\pi^2}{L^2} \quad \text{for } n = 1, 2, 3, \dots}$$

separation constant

family of solution X_n

$$\boxed{X_n(x) = \sin\left(\frac{n\pi x}{L}\right) \qquad n = 1, 2, 3, \dots}$$

NOTE 1: The $n=0$ term, we will not count

$\lambda_0 = 0 \longrightarrow X_0(L) = 0$

because $X_0(x) \equiv 0$

[illegible] a trivial solution, it doesn't

because $X_0(x) = 0$
but, this is a trivial solution, it doesn't tell us anything about $u(x,t)$

<u>NOTE 2:</u> $X_n'' = -\lambda_n X_n$

$\lambda_n = \frac{n^2\pi^2}{L^2}$ is called an <u>eigenvalue</u>

$X_n(x)$ is called an <u>eigenfunction</u>

Plug $\lambda_n = \frac{n^2\pi^2}{L^2}$ into the eqn for $T(t)$

$$T_n' + k\lambda_n T_n = 0$$

family of solutions

$$\boxed{T_n(t) = e^{-k\lambda_n t} = e^{-n^2\pi^2 kt/L^2} \quad n=1,2,3\ldots}$$

Recall, initially, we assumed that
$$u(x,t) = X(x)\,T(t)$$

Now we have a family of solution
$$u_n(x,t) = X_n(x)\,T_n(t)$$

$$\boxed{u_n(x,t) = e^{-kn^2\pi^2 t/L^2}\sin\left(\frac{n\pi x}{L}\right) \quad \text{for } n=1,2,3\ldots}$$

<u>NOTE:</u> each $u_n(x,t)$ satisfies: $\begin{cases} u_t = k u_{xx} \\ u(0,t) = u(L,t) = 0 \end{cases}$

By the Principle of Superposition, the general

By the Principle of Superposition, the general solution is a linear combination of these

$$u(x,t) = \sum_{n=1}^{\infty} b_n u_n(x,t)$$

coefficients b_n undetermined yet

$$\boxed{u(x,t) = \sum_{n=1}^{\infty} b_n e^{-kn^2\pi^2 t/L^2} \sin\left(\frac{n\pi x}{L}\right)}$$

Now impose the initial condition: $u(x,0) = f(x)$

$$u(x,0) = \sum_{n=1}^{\infty} b_n \cancel{e^{-kn^2\pi^2 0/L^2}}^{\,1} \sin\left(\frac{n\pi x}{L}\right) = f(x)$$

$$\underbrace{\sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) = f(x)}$$

This is a Fourier Sine Series of $f(x)$

$\rightarrow$ b_n are the Fourier coefficients

$$\boxed{b_n = \frac{2}{L}\int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx}$$

So the solution to the BVP (*) w/ homog BC is:

$$\boxed{\begin{gathered} u(x,t) = \sum_{n=1}^{\infty} b_n e^{-kn^2\pi^2 t/L^2} \sin\left(\frac{n\pi x}{L}\right) \\ b_n = \frac{2}{L}\int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx \end{gathered}}$$

Ex: Solve the BVP

$$\begin{cases} u_t = 3u_{xx} & \text{on} \quad 0 \le x \le \pi, \quad t > 0 \qquad L = \pi \\ u(0,t) = u(\pi,t) = 0 & \leftarrow \text{homog BC} \\ u(x,0) = 4\sin(2x) \end{cases}$$

$f(x) = 4\sin(2x) \qquad L = \pi \qquad k = 3$

$$u(x,t) = \sum_{n=1}^{\infty} b_n e^{-3n^2\cancel{\pi^2}t/\cancel{\pi^2}} \sin\left(\frac{n\cancel{\pi}x}{\cancel{\pi}}\right)$$

where

$$b_n = \frac{2}{\pi}\int_0^{\pi} \underbrace{4\sin(2x)\sin(nx)}_{\text{use the orthogonality of } \sin(nx)}\, dx$$

$$\int_{-\pi}^{\pi} \sin(mx)\sin(nx)\,dx = \begin{cases} \pi & \text{if } n = m \\ 0 & \text{if } n \neq m \end{cases}$$

so

$$\int_0^{\pi} \sin(2x)\sin(nx)\,dx = \begin{cases} \frac{\pi}{2} & \text{if } n = 2 \\ 0 & \text{if } n \neq 2 \end{cases}$$

$$b_n = \frac{2}{\pi}\cdot 4 \begin{cases} \frac{\pi}{2} & \text{if } n = 2 \\ 0 & \text{if } n \neq 2 \end{cases} \Rightarrow \quad \begin{array}{l} b_2 = 4 \\ b_n = 0 \text{ if } n \neq 2 \end{array}$$

$$u(x,t) = b_2 e^{-3\,2^2 t} \sin(2x)$$

$$\boxed{u(x,t) = 4e^{-12t}\sin(2x)}$$