Lecture 17

Section 9.5- Part 2

Heat Conduction & Separation of Variables

Announcements:

Online HW + A4 due Tues July 13

Midterm 1 grades released on Brightspace + Gradescope

Warm up:

Find the solution to the ODE:

$$\begin{cases} y''+\lambda y=0 & 0<x<L \\ y'(0)=0 \end{cases}$$

Ans: characteristic eqn $y=e^{rx}$

$$r^2+\lambda=0$$

roots $r=\pm i\sqrt{\lambda}$

general solution $y(x)=C_1\cos(\sqrt{\lambda}x)+C_2\sin(\sqrt{\lambda}x)$

impose BC $y'(0)=0$

$$y'(0)=0=\left[C_1(-\sqrt{\lambda})\sin(\sqrt{\lambda}x)+C_2\sqrt{\lambda}\cos(\sqrt{\lambda}x)\right]\Big|_{x=0}$$

$$0=C_1(-\sqrt{\lambda})\overset{0}{\cancel{\sin(0)}}+C_2\sqrt{\lambda}\overset{1}{\cos(0)}$$

$$\cdot\ 0=C_2$$

Solution $y(x)=C_1\cos(\sqrt{\lambda}x)$

last lecture we solved the Boundary Value Problem for the Heat Equation

$$\begin{cases} u_t=ku_{xx} & \text{on } 0\le x\le L,\ t>0 \\ u(0,t)=u(L,t)=0 & \text{(homogeneous BC)} \\ u(x,0)=f(x) & \text{(initial condition)} \end{cases}$$

We derived a solution using Separation of Variables

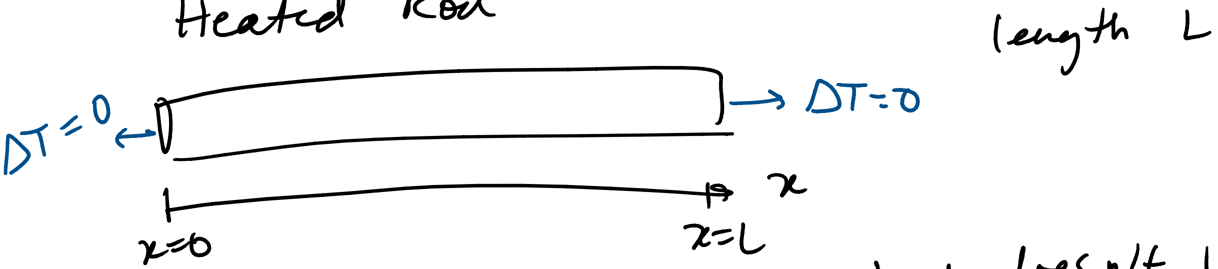
$$u(x,t)=\sum^{\infty} b_n e^{-kn^2\pi^2t/L^2}\sin\left(\frac{n\pi x}{L}\right)$$

$$\boxed{\begin{aligned} u(x,t) &= \sum_{n=1}^{\infty} b_n e^{-kn^2\pi^2 t/L^2} \sin\left(\frac{n\pi x}{L}\right) \\ \text{where } b_n &= \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx \end{aligned}}$$

Today, let's look at another BVP but with different Boundary Conditions.

I. Insulated Endpoint Conditions

Heated Rod



Endpoints are insulated — heat doesn't leave out the end

$\Delta T = 0$ in normal direction

Boundary Conditions:

$u_x(0,t) = u_x(L,t) = 0$

$\left[\frac{\partial u}{\partial x}(0,t) = \frac{\partial u}{\partial x}(L,t) = 0 \right.$ can also be written $\left. \right]$

Then, BVP:

$$\begin{cases} u_t = k u_{xx} & 0<x<L,\ t>0 \\ u_x(0,t) = u_x(L,t) = 0 & \text{(insulated endpts)} \\ u(x,0) = f(x) \end{cases}$$

To Solve $\longrightarrow$ Separation of Variables

Assume $u(x,t) = X(x)T(t)$

Assume $u(x,t) = X(x)\,T(t)$

plug into PDE

$$u_t = k\,u_{xx}$$

$$\frac{\partial}{\partial t}(XT) = k\frac{\partial^2}{\partial x^2}(XT)$$

$$XT' = kX''T$$

put all t terms oneside and x terms on other

$$\underbrace{\frac{T'}{kT}}_{\text{function of } t} = \underbrace{\frac{X''}{X}}_{\text{function of } x} = \text{constant} = \underbrace{-\lambda}_{\substack{\text{separation constant}\\ (\text{Assume } \lambda > 0)}}$$

separate out into ODE

$$\frac{X''}{X} = -\lambda \qquad\qquad \frac{T'}{kT} = -\lambda$$

$$X'' + \lambda X = 0 \qquad\qquad T' + \lambda k T = 0$$

NOTE: So far, everything has been the same as in the case for homogeneous B.C.

Now, let's apply the insulated endpoint BC

$$u_x(0,t) = 0 = X'(0)\,T(t) \rightarrow X'(0) = 0$$

$$u_x(L,t) = 0 = X'(L)\,T(t) \Rightarrow X'(L) = 0$$

So now, the ODE X

$$\begin{cases} X'' + \lambda X = 0 \end{cases}$$

NOTE: for homog BC.
$X(0) = X(L) = 0$

$$\begin{cases} X'' + \lambda X = 0 \\ X'(0) = X'(L) = 0 \end{cases}$$

$X(0) = X(L) = 0$

char eqn: $r^2 + \lambda = 0$

roots: $r = \pm i\sqrt{\lambda}$

general solution: $X = c_1 \cos(\sqrt{\lambda}x) + \cancel{c_2 \sin(\sqrt{\lambda}x)}$

apply end pt conditons

$$X'(0) = 0 = \left[c_1(-\sqrt{\lambda})\sin(\sqrt{\lambda}x) + c_2\sqrt{\lambda}\cos(\sqrt{\lambda}x)\right]\Big|_{x=0}$$

$$0 = -c_1\sqrt{\lambda}\,\overset{0}{\cancel{\sin(0)}} + c_2\sqrt{\lambda}\,\overset{1}{\cancel{\cos(0)}}$$

$$c_2 = 0$$

$$X'(L) = 0 = \left[-c_1\sqrt{\lambda}\sin(\sqrt{\lambda}x)\right]\Big|_{x=L}$$

$$0 = -c_1\sqrt{\lambda}\sin(\sqrt{\lambda}L)$$

$\sin\theta = 0$
when $\theta = n\pi$
$n = 1, 2, 3, \dots$

$$\sqrt{\lambda}L = n\pi$$

eigenvalues $\boxed{\lambda_n = \frac{n^2\pi^2}{L^2} \quad \text{for } n = 0, 1, 2, 3, \dots}$

eigenfunctions $\boxed{X_n = \cos\left(\frac{n\pi x}{L}\right)}$

NOTE: what if $n = 0$ $\quad \lambda_0 = 0$

$$\sin(\sqrt{\lambda_0}L) = \sin(0) = 0$$

then $X_0 = \cos(0) = 1$

this is a constant
nonzero, so
we want to keep it

Solve the T equation $\lambda_n = \frac{n^2\pi^2}{L^2}$

$$T_n' + k\lambda_n T = 0$$

Char eqn: $r + k\lambda_n = 0$ Roots $r = -k\lambda_n = -\frac{kn^2\pi^2}{L^2}$

$$\boxed{T_n = e^{-kn^2\pi^2 t/L} \qquad n = 0, 1, 2, \dots}$$

Family of solutions in n

$$u_n(x,t) = X_n T_n = e^{-kn^2\pi^2 t/L^2} \cos\left(\frac{n\pi x}{L}\right)$$

$n = 1, 2, 3, \dots$

when $n = 0$

$$u_0(x,t) = X_0 T_0 = e^{0} \cos(0) = 1$$

So, by the principle of superposition

$$u(x,t) = \sum_{n=0}^{\infty} a_n u_n(x,t)$$ also solves the ODE

For convenience, let's write

☆ $$\boxed{u(x,t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n e^{-kn^2\pi^2 t/L^2} \cos\left(\frac{n\pi x}{L}\right)}$$

NOTE: Let's choose $\frac{a_0}{2}$ to make this look like a Fourier Cosine Series

Let's plug in the initial condition:

$$u(x,0) = f(x) = \underbrace{\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \overset{1}{\cancel{e^{0}}} \cos\left(\frac{n\pi x}{L}\right)}_{\text{Fourier Cosine Series of } f(x)}$$

Fourier coefficients

$$a_0 = \frac{2}{L}\int_0^L f(x)\,dx$$

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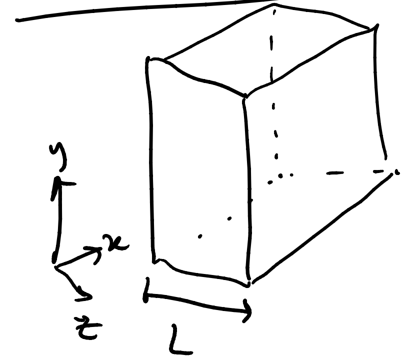
$$a_0 = \frac{2}{L}\int_0 f(x)\,d$$

$$☆ \quad a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx$$

☆ Solution to the Heat Eqn w/ insulated endpoints

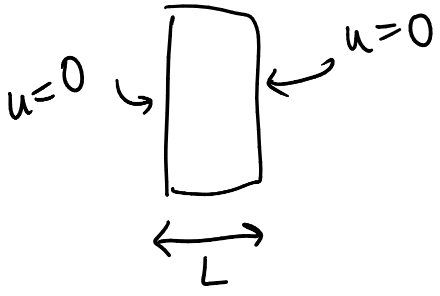
I. Heating a Slab:



Assume material is homogenous
- inside heats uniformly
- outside temp $T=0$ for all time
- thickness L

Dynamics are dominated by what happens in the z-direction

Look at a cross-section



no heat gradient in ↕

only have have heat gradient in ⟷

reduce this to a 1D equation

Ex: A copper slab is 4cm thick
($L = 4$cm $k = 1.15$ cm²/s)

Both faces are kept at 0°C
($u(0,t) = u(L,t) = 0$ homog. BC)

Initially the temp is 100°C
($u(x,0) = 100°$C initial condition)

Q: What is the temp at center 3s later?

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BVP:
$$\begin{cases} u_t = 1.15\, u_{xx} & 0 \leq x \leq 4, \quad t > 0 \\ u(0,t) = u(4,t) = 0 \\ u(x,0) = 100^\circ \end{cases}$$

solve for $u(x,t)$

Ex: Insulated Endpoints:

BVP
$$\begin{cases} 3u_t = u_{xx} & 0 \leq x \leq 2, \quad t > 0 \\ u_x(0,t) = u_x(2,t) = 0 \\ u(x,0) = \cos^2(2\pi x) \end{cases}$$

Here $L = 2$, $k = \frac{1}{3}$, $f(x) = \cos^2(2\pi x)$

Solution has the form:

$$u(x,t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n e^{-\frac{1}{3}n^2\pi^2 t/2} \cos\left(\frac{n\pi x}{2}\right)$$

Where:
$$a_0 = \frac{2}{2}\int_0^2 \cos^2(2\pi x)\,dx$$

$$a_n = \frac{2}{2}\int_0^2 \cos^2(2\pi x)\cos\left(\frac{n\pi x}{2}\right)dx$$

To evaluate these integrals, let's expand $f(x)$ using the double angle formula

$$\cos(2\theta) = 2\cos^2\theta - 1$$

$$\cos^2\theta = \frac{1}{2}\left[1 + \cos(2\theta)\right]$$

let $\theta = 2\pi x$ $\quad 2\theta = 4\pi x$

$$f(x) = \cos^2(2\pi x) = \frac{1}{2}\left[1 + \cos(4\pi x)\right]$$

$$a_0 = \int_0^2 \frac{1}{2}\left[1 + \cos(4\pi x)\right] dx$$

$$= \frac{1}{2}\left[x + \frac{\sin(4\pi x)}{4\pi} \right]_0^2 \qquad \boxed{a_0 = 1}$$

$$= \frac{1}{2}\left[2 + \cancelto{0}{\frac{\sin(8\pi)}{4\pi}} - 0 - \cancelto{0}{\sin(0)} \right] = \frac{2}{2} = 1$$

$$a_n = \int_0^2 \frac{1}{2}\left[1 + \cos(4\pi x)\right] \cos\left(\frac{n\pi x}{2}\right) dx$$

$$= \frac{1}{2}\left[\int_0^2 \cos\left(\frac{n\pi x}{2}\right) dx + \int_0^2 \overbrace{\cos(4\pi x)\cos\left(\frac{n\pi x}{2}\right)}^{\text{want use orthogonality}} dx \right]$$

u substitution $4\pi x = 8u$ $\quad u = \frac{\pi x}{2}$ $\quad dx = \frac{2du}{\pi}$

$$= \frac{1}{2}\left[\left(\frac{\sin\left(\frac{n\pi x}{2}\right)}{\frac{n\pi}{2}} \right)_0^2 + \frac{2}{\pi}\int_0^{\pi} \cos(8u)\cos(nu)\, du \right]$$

orthogonality condition

$$\int_{-\pi}^{\pi} \cos(mu)\cos(nu)\, du = \begin{Bmatrix} \pi & \text{if } m=n \\ 0 & \text{if } m \neq n \end{Bmatrix}$$

$$\int_0^{\pi} \cos(8u)\cos(nu)\, du = \begin{Bmatrix} \frac{\pi}{2} & \text{if } n=8 \\ 0 & \text{if } n \neq 8 \end{Bmatrix}$$

$$= \frac{1}{2}\left[\cancelto{0}{\frac{\sin(n\pi)}{\frac{n\pi}{2}}} - \cancelto{0}{\frac{\sin(0)}{\frac{n\pi}{2}}} + \frac{2}{\pi}\begin{Bmatrix} \frac{\pi}{2} & \text{if } n=8 \\ 0 & \text{if } n \neq 8 \end{Bmatrix} \right]$$

$$a_n = \frac{1}{2} \cdot \frac{2}{\pi} \cdot \begin{Bmatrix} \frac{\pi}{2} & \text{if } n=8 \\ 0 & \text{if } n \neq 8 \end{Bmatrix} = \begin{Bmatrix} \frac{1}{2} & \text{if } n=8 \\ 0 & \text{if } n \neq 8 \end{Bmatrix}$$

$$a_0 = 1 \qquad a_8 = \frac{1}{2} \qquad a_n = 0 \text{ if } n \neq 1, 8$$

So our solution:

$$u(x,t) = \frac{a_0}{2} + a_8 e^{-\frac{1}{3}(8)^2\pi^2 t/2^2} \cos\left(\frac{8\pi x}{2}\right)$$

$$-16\pi^2 t/3 \qquad \cos(4\pi x)$$

$$\boxed{u(x,t) = \frac{1}{2} + \frac{1}{2} e^{-16\pi^2 t/3} \cos(4\pi x)}$$

Homog. BC	Insulated BC
$u(0,t) = u(L,t) = 0$	$u_x(0,t) = u_x(L,t) = 0$
Fourier Sine Series	Fourier Cosine Series