Lecture 18

## Section 9.6 - Part 1

### Vibrating Strings & the 1D - Wave Equation

**Announcements:**

Online HW + A4 due Tues 7/13

Office Hours Today @ 2:30-3:30pm

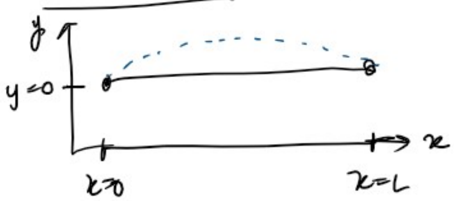
Masks still required in instructional spaces

Warm up: In Sec 9.5 we solved the heat eqn $u_t = k u_{xx}$ with 2 types of Boundary conditions:

| homogeneous BC | insulated endpoints |
|---|---|
| $\begin{cases} u_t = k u_{xx} \\ u(0,t) = u(L,t) = 0 \\ u(x,0) = f(x) \end{cases}$ | $\begin{cases} u_t = k u_{xx} \\ u_x(0,t) = u_x(L,t) = 0 \\ u(x,0) = f(x) \end{cases}$ |
| $u(x,t) = \sum_{n=1}^{\infty} b_n e^{-kn^2\pi^2 t/L^2} \sin\left(\frac{n\pi x}{L}\right)$ | $u(x,t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n e^{-kn^2\pi^2 t/L^2} \cos\left(\frac{n\pi x}{L}\right)$ |
| Fourier Sine Series | Fourier Cosine Series |

GOAL: Solve the 1D Wave Eqn

**I. Vibrating Strings:**




string length $L$

fixed at endpoints

@ $t=0$, pluck the string and measure the displacement $y(x,t)$

Model the displacement with the 1D-Wave Eqn:

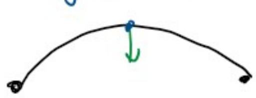
$$y_{tt} = a^2 y_{xx}$$

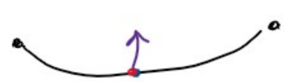
where $a^2 = \frac{\text{tension}}{\text{density}} > 0$ (constant)

physical intuition:

$y_{tt}$ = acceleration of string

$y_{xx}$ - curvature of string (in space)





$y_{xx} < 0$

so $y_{tt} = a^2 y_{xx} < 0$

string accelerates downward

$y_{xx} > 0$

$y_{tt} = a^2 y_{xx} > 0$

accelerates upwards

intuition: string wants to restore to equilibrium

**Boundary Value Problem:**

$0 \le x \le L, \ t > 0$

$\{ y_{tt} = a^2 y_{xx}$

## Boundary Value Problem:

$$(*)\begin{cases} y_{tt} = a^2 y_{xx} & 0 \le x \le L,\ t > 0 \\ y(0,t) = y(L,t) = 0 & \text{(fixed endpoints)} \\ y(x,0) = f(x) & \text{(initial displacement)} \\ y_t(x,0) = g(x) & \text{(initial velocity)} \end{cases}$$

NOTE: We can make problem easier to solve by separating into 2 BVPs and then add their solutions

**Problem A**

$$\begin{cases} y_{tt} = a^2 y_{xx} \\ y(0,t) = y(L,t) = 0 \\ y(x,0) = f(x) \\ y_t(x,0) = 0 \end{cases}$$

**Problem B**

$$y_{tt} = a^2 y_{xx}$$
$$y(0,t) = y(L,t) = 0$$
$$y(x,0) = 0$$
$$y_t(x,0) = g(x)$$

Let $y_A(x,t)$ solve Problem A

$y_B(x,t)$ solve Problem B

Then, since the wave equation is linear

$y(x,t) = y_A(x,t) + y_B(x,t)$ solves $(*)$

## II. Problem A:

$$\begin{cases} y_{tt} = a^2 y_{xx} \\ y(0,t) = y(L,t) = 0 \\ y(x,0) = f(x) \\ y_t(x,0) = 0 \end{cases}$$

Solve using Separation of Variables

assume: $y_A(x,t) = X(x)\,T(t)$

plug into the PDE

$$y_{tt} = a^2 y_{xx}$$
$$\frac{\partial^2}{\partial t^2}(XT) = a^2 \frac{\partial^2}{\partial x^2}(XT)$$
$$XT'' = a^2 X'' T$$

put all $x$ terms on one side, etc

$$\frac{T''}{a^2 T} = \frac{X''}{X} = -\lambda \quad \text{(separation constant)}$$

split into 2 ODE

$$X'' + \lambda X = 0 \qquad\qquad T'' + a^2 \lambda T = 0$$

$$X'' + \lambda X = 0 \qquad\qquad T'' + a^2 \lambda T = 0$$

Impose endpoint BC

$$y(0,t) = 0 = X(0)\,T(t) \longrightarrow X(0) = 0$$

$$y(L,t) = 0 = X(L)\,T(t) \longrightarrow X(L) = 0$$

Endpoint value problem:

$$\begin{cases} X'' + \lambda X = 0 \\ X(0) = X(L) = 0 \end{cases}$$

same ODE for $X(x)$ as the heat eqn w/ homog. BC

$$\boxed{\begin{aligned} X_n &= \sin\left(\frac{n\pi x}{L}\right) \\ \lambda_n &= \frac{n^2\pi^2}{L^2} \qquad n = 1, 2, 3, \dots \end{aligned}}$$

derived in Sec 9.5 - Part 1

Now, let's solve the $T$ equation

$$T_n'' + a^2 \lambda_n T_n = 0 \qquad \left(\text{heat eqn: } T_n' + k\lambda_n T = 0\right)$$

$$T_n'' + \frac{a^2 n^2 \pi^2}{L^2} T_n = 0$$

Are there any BC we can impose?

$$y(x,0) = f(x) = X_n(x)\,T_n(0) \longrightarrow \text{save for later}$$

$$y_t(x,0) = 0 = X_n(x)\,T_n'(0) \longrightarrow T_n'(0) = 0$$

IVP

$$\begin{cases} T_n'' + \dfrac{a^2 n^2 \pi^2}{L^2} T_n = 0 \\ T_n'(0) = 0 \end{cases}$$

characteristic eqn: $\quad r^2 + \dfrac{a^2 n^2 \pi^2}{L^2} = 0$

roots: $\quad r = \pm i\left(\dfrac{an\pi}{L}\right)$

general solution: $\quad T_n(t) = C_1 \cos\left(\dfrac{an\pi t}{L}\right) + \cancelto{0}{C_2 \sin\left(\dfrac{an\pi t}{L}\right)}$

Initial condition

$$T_n'(0) = 0 = \left[ C_1\left(-\frac{an\pi}{L}\right)\cancelto{0}{\sin\left(\frac{an\pi t}{L}\right)} + C_2\left(\frac{an\pi}{L}\right)\cancelto{1}{\cos\left(\frac{an\pi t}{L}\right)} \right]\Bigg|_{t=0}$$

$$0 = C_2\left(\frac{an\pi}{L}\right) \longrightarrow C_2 = 0$$

$$0 = c_2\left(\frac{an\pi}{L}\right) \longrightarrow c_2 = 0$$

$$\boxed{T_n(t) = \cos\left(\frac{an\pi t}{L}\right) \qquad n = 1, 2, 3, \ldots}$$

So for each $n$

$$y_n(x,t) = T_n X_n = \cos\left(\frac{n\pi a t}{L}\right)\sin\left(\frac{n\pi x}{L}\right)$$

By the Principle of Superposition

$$y_A(x,t) = \sum_{n=1}^{\infty} A_n y_n(x,t)$$

$$\star\ \boxed{y_A(x,t) = \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi a t}{L}\right)\sin\left(\frac{n\pi x}{L}\right)}$$

Impose last condition: $y(x,0) = f(x)$

$$y_A(x,0) = f(x) = \underbrace{\sum_{n=1}^{\infty} A_n \cancelto{1}{\cos(0)} \sin\left(\frac{n\pi x}{L}\right)}_{\text{this is a Fourier Sine Series}}$$

Fourier coefficients

$$\star\ \boxed{A_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx}$$

Solution to Problem A.

## III. Problem B:

$$\begin{cases} y_{tt} = a^2 y_{xx} \\ y(0,t) = y(L,t) = 0 \\ y(x,0) = 0 \\ y_t(x,0) = g(x) \end{cases}$$

Solve using Sep. of. Vars

$$y(x,t) = X(x)T(t)$$

plug into PDE

$$XT'' = a^2 X'' T$$

$$\frac{T''}{a^2 T} = \frac{X''}{X} = -\lambda$$

2 ODES

$$X'' + \lambda X = 0 \qquad\qquad T'' + a\lambda T = 0$$

Evaluate the BC

$$y(0,t) = 0 = X(0)T(t) \longrightarrow X(0) = 0$$

$$X(L)T(t) \longrightarrow X(L) = 0$$

Evaluate ...

$y(0,t) = 0 = X(0)T(t) \longrightarrow X(0)=0$

$y(L,t) = 0 = X(L)\,T(t) \longrightarrow X(L)=0$

$y(x,0) = 0 = X(x)\,T(0) \longrightarrow T(0)=0$

$y_t(x,0) = g(x) = X(x)\,T'(0) \longrightarrow$ save for later

$$\begin{cases} X'' + \lambda X = 0 \\ X(0) = X(L) = 0 \end{cases}$$

same as in Problem A

$$\boxed{\begin{gathered}\lambda_n = \frac{n^2\pi^2}{L^2} \\ X_n = \sin\left(\frac{n\pi x}{L}\right) \\ \text{for } n=1,2,3,\dots\end{gathered}}$$

$$\begin{cases} T'' + a^2\lambda T = 0 \\ T(0) = 0 \end{cases}$$

(same gen. soln as Problem A)

general solution

$$T = C_1\cancel{\cos}\left(\frac{n\pi a t}{L}\right) + C_2\sin\left(\frac{n\pi a t}{L}\right)$$

impose initial cond:

$$T(0) = 0 = C_1\cos(0) + C_2\cancel{\sin(0)}^{0}$$

$$C_1 = 0$$

$$\boxed{T_n = \sin\left(\frac{n\pi a t}{L}\right)}$$

we get

$$y_n(x,t) = \sin\left(\frac{n\pi a t}{L}\right)\sin\left(\frac{n\pi x}{L}\right)$$

By superposition

$$\boxed{y_B(x,t) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi a t}{L}\right)\sin\left(\frac{n\pi x}{L}\right)}$$

Impose last BC

$$y_t(x,0) = g(x)$$

$$g(x) = y_t(x,0) = \left.\frac{\partial y_B}{\partial t}\right|_{t=0} = \sum_{n=1}^{\infty} B_n \frac{\partial}{\partial t}\left[\sin\left(\frac{n\pi a t}{L}\right)\right]\sin\left(\frac{n\pi x}{L}\right)\Bigg|_{t=0}$$

$$= \sum_{n=1}^{\infty} B_n\left[\left(\frac{n\pi a}{L}\right)\cos\left(\frac{n\pi a t}{L}\right)\right]\sin\left(\frac{n\pi x}{L}\right)\Bigg|_{t=0}$$

$$g(x) = \underbrace{\sum_{n=1}^{\infty} B_n\left(\frac{n\pi a}{L}\right)\cancel{\cos(0)}^{1}\sin\left(\frac{n\pi x}{L}\right)}_{\text{Fourier Sine Series}}$$

Another way to show this

$$B_n\left(\frac{n\pi a}{L}\right) = b_n = \frac{2}{L}\int_0^L g(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

$$B_n = \frac{2}{n\pi a}\int_0^L g(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

Let's express $g(x)$ in terms of its F.S.S.

$$g(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) = \sum_{n=1}^{\infty} B_n\left(\frac{n\pi a}{L}\right)\sin\left(\frac{n\pi x}{L}\right) = \left.\frac{\partial y_B}{\partial t}\right|_{t=0}$$

$$g(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) = \sum_{n=1} B_n\left(\frac{n\pi a}{L}\right)\sin\left(\frac{\cdots}{L}\right) \quad \text{at } t$$

$$\sum_{n=1}^{\infty} \underbrace{\left[b_n - B_n\left(\frac{n\pi a}{L}\right)\right]}_{\text{must equal zero}} \sin\left(\frac{n\pi x}{L}\right) = 0$$

$$b_n - B_n\left(\frac{n\pi a}{L}\right) = 0$$

$$\frac{L b_n}{n\pi a} = B_n$$

$$b_n = \frac{2}{L}\int_0^L g(x)\sin\left(\frac{n\pi x}{L}\right)dx \leftarrow \text{Fourier coeff.}$$

$$B_n = \frac{L b_n}{n\pi a} = \left(\frac{\cancel{L}}{n\pi a}\right)\frac{2}{\cancel{L}}\int_0^L g(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

$$\boxed{B_n = \frac{2}{n\pi a}\int_0^L g(x)\sin\left(\frac{n\pi x}{L}\right)dx}$$

IV. Full solution to the BVP

$$y(x,t) = y_A(x,t) + y_B(x,t)$$

$$y(x,t) = \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi a t}{L}\right)\sin\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi a t}{L}\right)\sin\left(\frac{n\pi x}{L}\right)$$

$$A_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

$$B_n = \frac{2}{n\pi a}\int_0^L g(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

★ Next Class:

- example of solving a vibrating strings problem
- Separation of Variables
- Separation Constant → $\lambda$