Lecture 19

Section 9.6 – Part 2

Vibrating Strings & the 1D Wave Eqn

Announcements:

Online HW + A4 due Today @ 11:59pm

Office Hours Today @ 2:30–3:30pm on zoom

Warm up:

Evaluate the following integral

$$\int_0^1 \sin(3\pi x)\sin(n\pi x)\,dx = ?$$

Hint: you may want to use orthogonality of $\sin(nx)$:

$$\int_{-\pi}^{\pi} \sin(mx)\sin(nx)\,dx = \begin{cases} \pi & \text{if } m=n \\ 0 & \text{if } m\neq n \end{cases} \quad *$$

Ans: make a u substition

$u = \pi x \qquad n\pi x = nu$

$x = \frac{u}{\pi} \quad dx = \frac{du}{\pi} \qquad 3\pi x = 3u$

$$\int_0^1 \sin(3\pi x)\sin(n\pi x)\,dx = \frac{1}{\pi}\int_0^{\pi} \sin(3u)\sin(nu)\,du$$

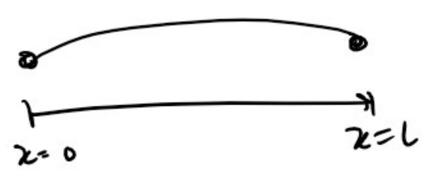
note interval is $[0,\pi]$ so integral is going to be $\frac{1}{2}$ of $*$

$$= \frac{1}{\pi}\begin{cases} \frac{\pi}{2} & \text{if } n=3 \\ 0 & \text{if } n\neq 3 \end{cases}$$

$$= \begin{cases} \frac{1}{2} & \text{if } n=3 \\ 0 & \text{if } n\neq 3 \end{cases}$$

I. Vibrating Strings:

Last class we wrote down a BVP for a vibrating string



string of length L fixed endpts

BVP:

$$\begin{cases} y_{tt} = a^2 y_{xx} & 0 \le x \le L,\ t>0 \quad \text{(wave eqn)} \\ y(0,t) = y(L,t) = 0 & \text{(fixed endpoints)} \\ y(x,0) = f(x) & \text{(initial displacement)} \\ & \text{(initial velocity)} \end{cases}$$

$$\begin{cases} y(x,0) = f(x) & \text{(initial ...)} \\ y_t(x,0) = g(x) & \text{(initial velocity)} \end{cases}$$

Use Separation of Variables to derive

$$\boxed{\begin{aligned} y(x,t) &= \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi a t}{L}\right)\sin\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi a t}{L}\right)\sin\left(\frac{n\pi x}{L}\right) \\ A_n &= \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx \\ B_n &= \frac{2}{n\pi a}\int_0^L g(x)\sin\left(\frac{n\pi x}{L}\right)dx \end{aligned}}$$

Solve an example

Ex:

$$\begin{cases} y_{tt} = y_{xx} & 0 \le x \le 1,\ t > 0 \\ y(0,t) = y(1,t) = 0 \\ y(x,0) = -3 \\ y_t(x,0) = 5\sin(3x) \end{cases}$$

Here: $a^2 = 1$, $L = 1$, $f(x) = -3$, $g(x) = 5\sin(3x)$

so
$$y(x,t) = \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi(1)t}{1}\right)\sin\left(\frac{n\pi x}{1}\right) + \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi(1)t}{(1)}\right)\sin\left(\frac{n\pi x}{1}\right)$$

Calculate the A_n

$$A_n = \frac{2}{1}\int_0^1 f(x)\sin\left(\frac{n\pi x}{1}\right)dx = 2\int_0^1 (-3)\sin(n\pi x)\,dx$$

$$\left(\ldots\cos(n\pi x)\ldots\right)\Big] = -6\Big[-\underbrace{\cos(n\pi)}_{(-1)^n} + \underbrace{\cos(0)}_{1}\Big]$$

$$= -6\left(-\frac{\cos(n\pi x)}{n\pi}\right)\Bigg|_0^1 = \frac{-6}{n\pi}\left[-\cancelto{(-1)^n}{\cos(n\pi)} + \cancelto{1}{\cos(0)}\right]$$

$$= \frac{-6}{n\pi}\left[1-(-1)^n\right] = \boxed{\begin{Bmatrix} \frac{-12}{n\pi} & \text{if } n \text{ odd} \\ 0 & \text{if } n \text{ even} \end{Bmatrix} = A_n}$$

Calculate the B_n

$$B_n = \frac{2}{n\pi(1)}\int_0^1 g(x)\sin\left(\frac{n\pi x}{l}\right)dx = \frac{2}{n\pi}\int_0^1 5\sin(3\pi x)\sin(n\pi x)\,dx$$

$$= \frac{10}{n\pi}\underbrace{\int_0^1 \sin(3\pi x)\sin(n\pi x)\,dx}_{= \begin{Bmatrix} \frac{1}{2} & \text{if } n=3 \\ 0 & \text{if } n \neq 3 \end{Bmatrix}}$$

integral from warm up

use orthogonality

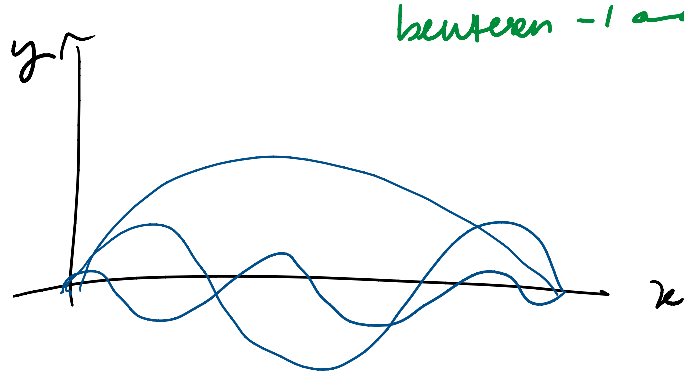
$$\boxed{B_n = \begin{Bmatrix} \frac{5}{n\pi} & \text{if } n=3 \\ 0 & \text{if } n\neq 3 \end{Bmatrix}}$$

So full solution is

$$\boxed{y(x,t) = \sum_{n \text{ odd}} \left(\frac{-12}{n\pi}\right)\underbrace{\cos(n\pi t)}_{\substack{\text{amplitude oscillate}\\ \text{between } -1 \text{ and } 1}}\sin(n\pi x) + \frac{5}{n\pi}\sin(3\pi t)\sin(3\pi x)}$$

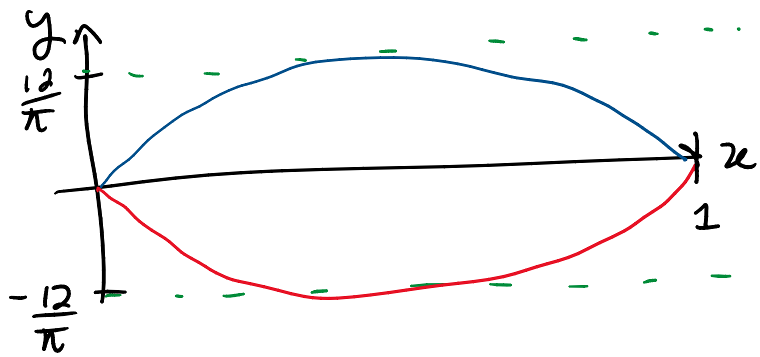
in x

$\cos(n\pi t)$ when n is odd



because $\frac{1}{n}$ in coeff, largest term $n=1$, $n=3, \ldots$

$\boxed{n=1}$ $y(x,t) \sim \frac{-12}{\pi}\cos(\pi t)\sin(\pi x)$

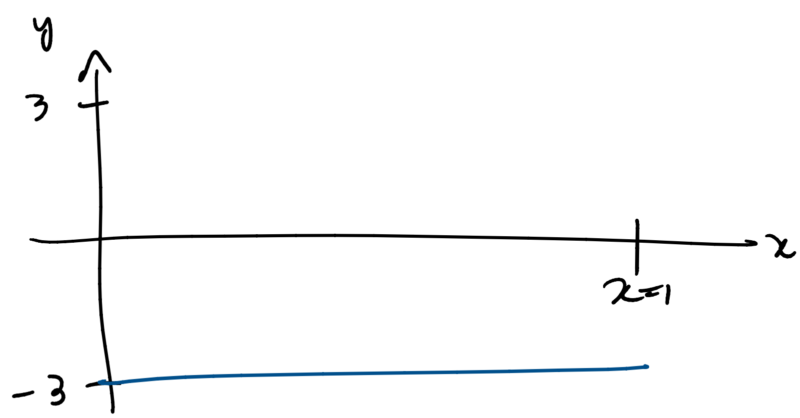


@ $t=0$

$y(x,0) = \frac{-12}{\pi}\sin(\pi x)$

@ $t=1$

$y(x,1) = \frac{-12}{\pi}(-1)\sin(\pi x)$



@ $t=0$

I. Separation of Variables:

works for the PDEs:

heat eqn $\qquad u_t = k u_{xx}$

wave eqn $\qquad u_{tt} = a^2 u_{xx}$

Method:

1. Assume function can be written as a product of 2 single-variable functions

$$u(x,t) = \underline{X}(x)\,T(t)$$

2. Plug into the PDE and collect x terms on one side and t terms on other

heat

$T' \qquad X''$

wave

$$\underline{T''} = \underline{X''}$$

neat

$$\frac{T'}{kT} = \frac{X''}{X} \qquad \frac{T''}{a^2T} = \frac{X''}{X}$$

3. Set both sides equal to a separation constant $(-\lambda)$ * we look at why

$$\frac{T'}{kT} = \frac{X''}{X} = -\lambda \qquad \frac{T''}{aT} = \frac{X''}{X} = -\lambda$$

4. Separate out into 2 ODEs

Heat:

$$X'' + \lambda X = 0$$
$$T' + \lambda k T = 0$$

wave

$$X'' + \lambda X = 0$$
$$T'' + a\lambda T = 0$$

5. Use the Boundary Conditions for $u(x,t)$ to get endpoint conditions for X (and sometimes T)

Heat:

homogeneous BC

$$X(0) = X(L) = 0$$

insulated endpoints

$$X'(0) = X'(L) = 0$$

wave eqn:

fixed string endpts:

$$X(0) = X(L) = 0$$

Problem A:

$$T'(0) = 0$$

Problem B:

$$T(0) = 0$$

6. Solve the ODEs. In the process, find value of λ

Heat:

$$T_n = e^{-kn^2\pi^2 t/L^2}$$
$$\lambda_n = \frac{n^2\pi^2}{L^2}$$

wave:

$$\lambda_n = \frac{n^2\pi^2}{L^2}$$
$$X_n = \sin\left(\frac{n\pi x}{L}\right)$$

homog. BC
$$X_n = \sin\left(\frac{n\pi x}{L}\right)$$

insulated
$$X_n = \cos\left(\frac{n\pi x}{L}\right)$$

Problem A:
$$T_n = \cos\left(\frac{n\pi a t}{L}\right)$$

Problem B
$$T_n = \sin\left(\frac{n\pi a t}{L}\right)$$

7. Then we have a family of solns:
$$u_n(x,t) = X_n(x)\, T_n(t)$$

Use the principle of superposition
$$u(x,t) = \sum_{n=1}^{\infty} c_n X_n(x)\, T_n(t)$$

8. Lastly, use the remaing BC to find the coefficients c_n

Heat Eqn: $u(x,0) = f(x)$ (assume homog. BC)

$$u(x,0) = \underbrace{\sum_{n=1}^{\infty} b_n \cancel{e^{0}}^{\,1} \sin\left(\frac{n\pi x}{L}\right)}_{\text{this is the F.S.S. of } f(x)} = f(x)$$

the b_n are the Fourier coefficients
$$b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

Q: Why do we choose $-\lambda$ as the separation constant?

Heat eqn: this time, choose $+\lambda$ as the separation constant

$$\begin{cases} u_t = k u_{xx} \\ u(0,t) = u(L,t) = 0 \\ u(x,0) = f(x) \end{cases}$$ Homog. BC

1. Let $u(x,t) = X(x)\,T(t)$

2. Plug into ODE: $\dfrac{T'}{kT} = \dfrac{X''}{X}$

3. Choose $+\lambda$ as the separation constant $(\lambda > 0)$

$$\frac{T'}{kT} = \frac{X''}{X} = +\lambda$$

4. Separate into 2 ODEs

$$\frac{X''}{X} = +\lambda \qquad\qquad \frac{T'}{kT} = +\lambda$$

$$X'' = \lambda X$$

$$X'' - \lambda X = 0 \qquad \text{sign is switched} \qquad T' - k\lambda T = 0$$

5. Find BC for X and T

homog BC $\longrightarrow$ $X(0) = X(L) = 0$

no BC in T

6. Solve the ODEs

$$\begin{cases} X'' - \lambda X = 0 \\ X(0) = X(L) = 0 \end{cases}$$

Assume $X = e^{rx}$

characteristic eqn: $r^2 - \lambda = 0$ $\longleftarrow$ real-valued roots

Assume λ ...

characteristic eqn: $r^2 - \lambda = 0$ ← real-valued roots

roots: $r = \pm\sqrt{\lambda}$ (before $r = \pm i\sqrt{\lambda}$)

general solution: (before we had sines and cosines)

$$\underline{X} = C_1 e^{\sqrt{\lambda} x} + C_2 e^{-\sqrt{\lambda} x}$$

plug in end point conditions

$$\underline{X}(0) = 0 = C_1 e^{0} + C_2 e^{-0} = C_1 + C_2$$

$\rightarrow C_2 = -C_1$

$$\rightarrow \underline{X}(x) = C_1 \left[e^{\sqrt{\lambda} x} - e^{-\sqrt{\lambda} x} \right]$$

$\sinh(\theta) = \dfrac{e^{\theta} - e^{-\theta}}{2}$

$$= C_1 \, 2 \sinh(\sqrt{\lambda}\, x)$$

2nd endpt: $\underline{X}(L) = 0 = \cancel{C_1}\left[e^{\sqrt{\lambda} L} - e^{-\sqrt{\lambda} L} \right]$

$$e^{\sqrt{\lambda} L}\left(e^{\sqrt{\lambda} L}\right) = \left(e^{-\sqrt{\lambda} L}\right) e^{\sqrt{\lambda} L}$$

$$\ln\left(e^{2\sqrt{\lambda} L}\right) = \ln(1)$$

$$2\sqrt{\lambda} L = 0$$

$$\rightarrow \sqrt{\lambda} = 0 \qquad \boxed{\lambda = 0}$$

$\Rightarrow\Leftarrow$

that says $\underline{X}(x) = C_1 e^{0x} - C_1 e^{0x} = 0$

if $\lambda = 0$, $\underline{X} \equiv 0$

$$u(x,t) = \underline{X}(x) T(t) = 0 \cdot T(t) = 0$$

$\Rightarrow\Leftarrow$

WANT: nonzero solutions to the heat eqn.

$\longrightarrow$ Need to choose separation constant $-\lambda$

<u>A</u>: Because we have the homogeneous BC

$$u(0,t) = u(L,t) = 0$$

and so we get the endpoint conditions

$$X(0) = X(L) = 0$$

so we want to choose separation constant $-\lambda$

so that $X'' + \lambda X = 0$

and $r = \pm i\sqrt{\lambda}$

and $X = c_1 \cos(\sqrt{\lambda}\, x) + c_2 \sin(\sqrt{\lambda}\, x)$

and then $X(L) = 0$ has <u>nonzero</u> solutions.