Lecture 1

★ Overview & Section 5.1

$\boxed{\text{MA 303}}$

Warm up: Which of the following equations are ODEs?
(check all that apply)

(a) $\frac{dy}{dt} = 3y$ ✓

(b) $y^2 = e^{7t} + 10t$ ✗

(c) $\frac{\partial y}{\partial t} = \frac{\partial^2 y}{\partial x^2}$ ✗ $y(t,x)$ PDE

(d) $y'' + 3y' + 7y = \cos(t)$ ✓

Instructor: Dr. Hood (kthood@purdue.edu)

Office Hours: MTW: 2:30–3:30pm on zoom*

(* use zoom tab in Brightspace)

MA 303: Differential Eqns & PDEs

ODEs: (Ordinary Diff. Eqn) relates a function of one variable with its derivatives

Ex: $\frac{dy}{dt} = 3y$ $\qquad y'' + 3y' + 7y = \cos(t)$

PDEs: (Partial Differential Eqn) relates a function of multiple variables with its partial derivatives

Ex: $\frac{\partial y}{\partial t} = \frac{\partial^2 y}{\partial x^2}$ $\qquad$ Here $y(t,x)$

Today: Chapter 5

GOAL: Solve linear systems of ODEs

Recall: We can rewrite a higher order ODE as a 1st order system

Recall: We can [illegible]
a 1st order system

Ex: $y'' + 3y' + 7y = 0$

System: Let $x_1 = y$ $\qquad x_2 = y'$

$$x_1' = y' = x_2$$

$$x_2' = y'' = -3y' - 7y = -3x_2 - 7x_1$$

System: $\begin{cases} x_1' = x_2 \\ x_2' = -3x_2 - 7x_1 \end{cases}$

Matrix Form

$$\underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}'}_{\underline{x}\text{ vector}} = \underbrace{\begin{bmatrix} 0 & 1 \\ -7 & -3 \end{bmatrix}}_{\underline{\underline{A}}\text{ coefficient matrix}} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad \underline{x}' = \underline{\underline{A}}\,\underline{x}$$

Section 5.1 — review of linear algebra + systems

I. Matrices:

A $m \times n$ matrix $\underline{\underline{A}}$

$$\underline{\underline{A}} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

$a_{\text{row, col}}$

m rows

n columns

Notation: $\underline{\underline{A}}$ is a matrix $m \times n$ — uppercase letters

$\underline{x}$ is a vector $(n \times 1)$ — lowercase letters $\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$

(a) matrix addition

$$\begin{bmatrix}1&2\\3&4\end{bmatrix}+\begin{bmatrix}1&2\\1&2\end{bmatrix}=\begin{bmatrix}1+1&2+2\\3+1&4+2\end{bmatrix}=\begin{bmatrix}2&4\\4&6\end{bmatrix}$$

(b) scalar multiplication

$$2\begin{bmatrix}1&2\\3&4\end{bmatrix}=\begin{bmatrix}2\cdot1&2\cdot2\\2\cdot3&2\cdot4\end{bmatrix}=\begin{bmatrix}2&4\\6&8\end{bmatrix}$$

scalar

multiply each element by 2

(c) Transpose of a matrix $\underline{A}^T$

row ⟶ column

$$\underset{(2\times2)}{\underline{A}}=\begin{bmatrix}1&2\\3&4\end{bmatrix}\qquad \underset{(2\times2)}{\underline{A}^T}=\begin{bmatrix}1&3\\2&4\end{bmatrix}$$

$$\underline{\underline{B}}=\underset{(2\times3)}{\begin{bmatrix}1&2&3\\4&5&6\end{bmatrix}}\qquad \underline{\underline{B}}^T=\begin{bmatrix}1&4\\2&5\\3&6\end{bmatrix}\ (3\times2)$$

II Matrix Multiplication

$\underline{\underline{A}}\ (m\times p)$ $\qquad$ $\underline{\underline{B}}\ (p\times n)$

Then $\underline{\underline{A}}\,\underline{\underline{B}}$ is defined ⟶ $(m\times n)$

$$\underline{\underline{A}}=\begin{bmatrix}1&2&3\\4&5&6\end{bmatrix}\ (2\times3)\qquad \underline{\underline{B}}=\begin{bmatrix}1&1\\2&1\\1&1\end{bmatrix}\ (3\times2)$$

$$\underset{(2\times3)(3\times2)}{\underline{\underline{A}}\,\underline{\underline{B}}}=\begin{bmatrix}1&2&3\\4&5&6\end{bmatrix}\begin{bmatrix}1&1\\2&1\\1&1\end{bmatrix}=\underset{(2\times2)}{\begin{bmatrix}\ &\ \\ \ &\ \end{bmatrix}}$$

$$=\begin{bmatrix}1\cdot1+2\cdot2+3\cdot1 & 1\cdot1+2\cdot1+3\cdot1\\ 4\cdot1+5\cdot2+6\cdot1 & 4\cdot1+5\cdot1+6\cdot1\end{bmatrix}=\begin{bmatrix}8&6\\20&15\end{bmatrix}=\underset{(2\times2)}{\underline{\underline{A}}\,\underline{\underline{B}}}$$

Ex: $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$ $\qquad C = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

$\underset{(2\times 3)\ (2\times 2)}{A\,C} = \times$ NOT Defined

$\times$

✓

$\underset{(2\times 2)(2\times 3)}{C\,A} = \begin{bmatrix} & & \\ & & \end{bmatrix} (2\times 3)$ this is defined

Poll: $\underset{}{A} = \overset{(3\times 3)}{\begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}}$ $\qquad B = \overset{(2\times 3)}{\begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \end{bmatrix}}$

Which of the following matrix mult. is valid?

$\times$ (a) AB $(3\times3)(2\times3)$ (c) neither

✓ (b) BA $(2\times3)(3\times3)$ (d) both $\times$

III. Determinant:

If A is a 2×2 matrix then its determinant

$$\det(A) = |A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

Ex: $\det\begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} = 1\cdot 3 - 2\cdot 4 = 3 - 8 = -5$

For determinant of bigger square matrices, we use Expansion by Minors

If A is an $n\times n$ matrix

... $(n-1)\times(n-1)$ matrix obtained by

If $\underline{A}$ is an $n \times n$ matrix

Let $\underline{A}_{ij}$ be the $(n-1)\times(n-1)$ matrix obtained by deleting the ith row and jth column

$$\det(\underline{A}) = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \det(\underline{A}_{ij})$$

Ex: $\underline{A} = \begin{bmatrix} 3 & 1 & -2 \\ 4 & 2 & 1 \\ -2 & 3 & 5 \end{bmatrix}$ let $i=1$

$$\det(\underline{A}) = (-1)^{1+1} a_{11} \det(\underline{A}_{11}) + (-1)^{1+2} a_{12} \det(\underline{A}_{12}) + (-1)^{1+3} a_{13} \det(\underline{A}_{13})$$

$$= (-1)^2 (3) \det\begin{bmatrix} 2 & 1 \\ 3 & 5 \end{bmatrix} + (-1)^3 (1) \det\begin{bmatrix} 4 & 1 \\ -2 & 5 \end{bmatrix} + (-1)^4 (-2) \det\begin{bmatrix} 4 & 2 \\ -2 & 3 \end{bmatrix}$$

$$= 3(2\cdot 5 - 1\cdot 3) + (-1)(4\cdot 5 - 1(-2)) + (-2)(4\cdot 3 - 2(-2))$$

$$\boxed{\det(\underline{A}) = -33}$$

IV, Matrix-Valued Functions:

Def: A matrix-valued function is a matrix in which each element is a function of t

Ex: $\underline{x}(t) = \begin{bmatrix} t \\ 3t^2 \end{bmatrix}$ $\underline{A}(t) = \begin{bmatrix} \sin(t) & 7 \\ 0 & 8t^2+1 \end{bmatrix}$

matrix fcn $\underline{A}(t)$ is continuous

Def: We say that a matrix fcn $\underline{A}(t)$ is continuous at a point t if each of its elements is continuous

Def: The derivative of a matrix fcn is defined by elementwise differentiation

Ex: $\underline{x}(t) = \begin{bmatrix} t \\ t^2 \\ e^{-t} \end{bmatrix}$ $\quad \underline{x}'(t) = \frac{d\underline{x}}{dt} = \begin{bmatrix} 1 \\ 2t \\ -e^{-t} \end{bmatrix}$

sum and product rules

$$\frac{d}{dt}\left(\underline{\underline{A}}(t) + \underline{\underline{B}}(t)\right) = \underline{\underline{A}}' + \underline{\underline{B}}'$$

$$\frac{d}{dt}\left(\underline{\underline{A}}\,\underline{\underline{B}}\right) = \underline{\underline{A}}\,\underline{\underline{B}}' + \underline{\underline{A}}'\underline{\underline{B}}$$ HW problem to show this

$$\frac{d}{dt}\left(c\,\underline{\underline{A}}(t)\right) = c\,\underline{\underline{A}}'$$ c is a scalar const (doesn't depend on t)

V First Order Linear Systems:

system of ODEs $\longrightarrow$ matrix form

$$\underset{(n\times1)}{\frac{d\underline{x}}{dt}} = \underset{(n\times n)(n\times 1)}{\underbrace{\underline{\underline{P}}(t)}_{\text{coefficient matrix}}\,\underline{x}} + \underset{(n\times1)}{\underline{f}(t)}$$

Ex: 1st order system

$x_1' = 4x_1 - 3x_2$

$x_2' = 6x_1 - 7x_2$

matrix form

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}' = \begin{bmatrix} 4 & -3 \\ 6 & -7 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Verify $\underline{x}(t) = \begin{bmatrix} 3e^{2t} \\ 2t \ldots \end{bmatrix}$ is a soln

Verify $\underline{x}(t) = \begin{bmatrix} 3e^{2t} \\ 2e^{2t} \end{bmatrix}$ is a soln

$$\underline{x}' \overset{?}{=} \begin{bmatrix} 4 & -3 \\ 6 & -7 \end{bmatrix} \underline{x}$$

VI. Associated Homogeneous Eqn

when $\underline{f}(t) = 0$ the eqn is homogeneous

$$(*) \qquad \underline{x}' = \underline{\underline{P}}(t)\,\underline{x}$$

If $\underline{\underline{P}}(t)$ is an $n \times n$ matrix, then we expect $(*)$ to have n linearly independent solutions

$$\underline{x}_1(t),\ \underline{x}_2(t),\ \dots,\ x_n(t)$$

Since the system is linear, the Principle of Superposition applies

$$\underline{x}(t) = c_1 \underline{x}_1(t) + c_2 \underline{x}_2(t) + \dots + c_n \underline{x}_n(t)$$

where $c_1, c_2, \dots, c_n$ are scalar constants

then $\underline{x}(t)$ also solves $(*)$

Ex: $\underline{x}' = \begin{bmatrix} 4 & -3 \\ 6 & -7 \end{bmatrix} \underline{x}$

has solutions $\underline{x}_1(t) = \begin{bmatrix} 3e^{2t} \\ 2e^{2t} \end{bmatrix}$ $\quad \underline{x}_2(t) = \begin{bmatrix} e^{-5t} \\ 3e^{-5t} \end{bmatrix}$

Then the general solution is

$$\underline{x}(t) = c_1 \underline{x}(t) + c_2 \underline{x}_2(t)$$

$$= c_1 \begin{bmatrix} 3e^{2t} \\ \end{bmatrix} + c_2 \begin{bmatrix} e^{-5t} \\ \end{bmatrix}$$

$$\underline{x}(t) = c_1 \underline{x}_1 \ldots$$

$$= c_1 \begin{bmatrix} 3e^{2t} \\ 2e^{2t} \end{bmatrix} + c_2 \begin{bmatrix} e^{-5t} \\ 3e^{-5t} \end{bmatrix}$$

VII <u>Linear Independence</u>: Wronskian

If $\underline{x}_1, \ldots \underline{x}_n$ are solutions to ODE $(*)$

the <u>Wronskian</u> is

$$W(t) = \det \begin{bmatrix} | & | & & | \\ \underline{x}_1(t) & \underline{x}_2(t) & \cdots & x_n(t) \\ | & | & & | \end{bmatrix} \quad (n \times n)$$

If $W(t) \neq 0$ then are linearly independent

<u>Ex</u>:

$$W(t) = \det \begin{bmatrix} 3e^{2t} & e^{-5t} \\ 2e^{2t} & 3e^{-5t} \end{bmatrix} = (3e^{2t})(3e^{-5t}) - (2e^{2t})(e^{-5t})$$

$$= 9e^{-3t} - 2e^{-3t}$$

$$= 7e^{-3t} \neq 0$$

so $\underline{x}_1$ and $\underline{x}_2$ are <u>linearly independent</u>

☆ <u>Summary</u>:

- $\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = ad - bc$
- use <u>expansion by minors</u> for determinants of $n \times n$ matrices
- vector valued functions — take derivatives elementwise

- matrix-valued functions — take derivatives elementwise

- Homogeneous Linear Systems

$$\underline{x}' = \underline{\underline{P}}(t)\underline{x} \qquad \text{where } \underline{\underline{P}}(t) \text{ is } n \times n$$

have n linearly independent solutions

$$\underline{x}_1(t), \underline{x}_2(t), \ldots, \underline{x}_n(t)$$

use the <u>Principle of superposition</u> to get the general solution:

$$\underline{x}(t) = c_1 \underline{x}_1(t) + \ldots + c_n \underline{x}_n(t)$$

- Use the Wronskian to determine linear independence.