Lecture 20

☆Section 9.7 - Part 1

Steady-State Temperature & Laplace's Equation

Announcements:

Office Hours Today @ 2:30-3:30 pm on zoom

Final Exam: Friday Aug 6 @ 8am - 10am in CL50 224

Warm up:

So far, we have solved

1D Heat Eqn $u_t = k u_{xx}$

1D Wave Eqn $u_{tt} = a^2 u_{xx}$

In 2D (in space), they become:

2D Heat Eqn $u_t = k(u_{xx} + u_{yy})$

2D Wave Eqn $u_{tt} = a^2(u_{xx} + u_{yy})$

## I. Motivation:

Inspired by the 2D PDEs

$u_t = k(u_{xx} + u_{yy})$ and $u_{tt} = a^2(u_{xx} + u_{yy})$

$$u_{xx} + u_{yy} = \nabla^2 u = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) u$$

is called the Laplacian

GOAL: want to find the steady-state solution (when $u_t = 0$ or $u_{tt} = 0$)

means that $u(x,y,t)$ is no longer changing in time.

2D Heat Eqn Steady State: $u_t = 0 = k(u_{xx} + u_{yy})$

2D Wave Eqn Steady State: $u_{tt} = 0 = a^2(u_{xx} + u_{yy})$

Both of these satisfy Laplace's Equation

$$\boxed{u_{xx} + u_{yy} = 0}$$

## II. Dirichlet Problems:

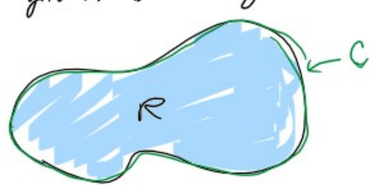
Before we were solving Boundary Value Problems

Today we will solve the Dirichlet Problem:

find a solution to Laplace's equation in a region $R$ with given boundary values on the curve $C$

region $R$ (2D)

curve $C$ (1D)




Dirichlet problem:

$$\begin{cases} u_{xx} + u_{yy} = 0 & \text{for } (x,y) \text{ in region } R \\ u(x,y) = f(x,y) & \text{for } (x,y) \text{ in curve } C \end{cases}$$

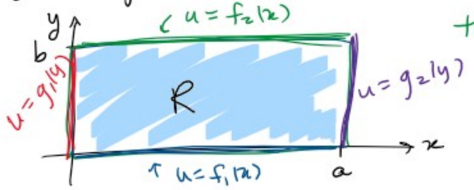
NOTE: If $f(x,y)$ and the curve $C$ are "nice" then there is a unique solution $u(x,y)$

## III. Rectangular Domain $R$:

Let region $R$ be a rectangle

## III. Rectangular Domain R:

Let region $R$ be a rectangle




the curve $C$ is a set of 4 straight lines

**Dirichlet Problem on the Rectangle:**

$$\begin{cases} u_{xx}+u_{yy}=0 & 0\le x\le a,\ 0\le y\le b \\ u(x,0)=f_1(x) \\ u(x,b)=f_2(x) \\ u(0,y)=g_1(y) \\ u(a,y)=g_2(y) \end{cases}$$

To Solve this, use Separation of Variables

NOTE: Your HW involves deriving solutions using separation of variables

Ex: $$\begin{cases} u_{xx}+u_{yy}=0 & 0\le x\le a,\ 0\le y\le b \\ u(0,y)=u(a,y)=u(x,b)=0 \\ u(x,0)=f(x) \end{cases}$$

Here $f_1(x)=f(x)$ $\quad f_2=g_1=g_2\equiv 0$

Separation of Variables

assume $u(x,y)=X(x)\,Y(y)$

plug into the PDE

$u_{xx}+u_{yy}=0$

$X''Y+XY''=0$

separate like terms

$$\frac{X''}{X}=-\frac{Y''}{Y}=-\lambda$$

notice minus sign

separation constant (start w/ $-\lambda$ may need to change to $+\lambda$ later)

split into 2 ODEs

$$\frac{X''}{X}=-\lambda \qquad\qquad -\frac{Y''}{Y}=-\lambda$$

$$X''+\lambda X=0 \qquad\qquad Y''-\lambda Y=0$$

opposite signs in the ODEs

Let's evaluate the Boundary Conditions

$u(0,y)=0=X(0)Y(y) \quad\rightarrow\quad X(0)=0$

$u(a,y)=0=X(a)Y(y) \quad\rightarrow\quad X(a)=0$

$u(x,0)=f(x)=X(x)Y(0)$ $\quad$ save for later

$u(x,b)=0=X(x)Y(b) \quad\rightarrow\quad Y(b)=0$

So now our ODEs become:

$$\begin{cases} X''+\lambda X=0 \\ X(0)=X(a)=0 \end{cases} \qquad \begin{cases} Y''-\lambda Y=0 \\ Y(b)=0 \end{cases}$$

So now our ...

$$\begin{cases} X'' + \lambda X = 0 \\ X(0) = X(a) = 0 \end{cases} \qquad \begin{cases} Y'' - \lambda Y = 0 \\ Y(b) = 0 \end{cases}$$

Q: Should we change the separation constant from $-\lambda$ to $+\lambda$ ?

We have $+\lambda$ in ODE and 2 endpt conditions

We have $-\lambda$ in ODE, 1 endpt condition

$+\lambda \to X$ is sines and cosines

$-\lambda \to Y$ is exponentials $e^{\sqrt{\lambda}y}$ and $e^{-\sqrt{\lambda}y}$

the BC will be easily satisfied ✓

but, there's only 1 BC so this will be OK ✓

Yes, let's keep $-\lambda$ as the separation constant

Rule: Want $+\lambda$ on the ODE with 2 BC

Solve the $X$ equation

$$\begin{cases} X'' + \lambda X = 0 \\ X(0) = X(a) = 0 \end{cases}$$

guess $X = e^{rx}$

char. eqn: $r^2 + \lambda = 0$

roots: $r = \pm i\sqrt{\lambda}$

gen soln: $X = c_1 \cancel{\cos}(\sqrt{\lambda}x) + c_2 \sin(\sqrt{\lambda}x)$

impose BC

$X(0) = 0 = c_1 \cos(0) + c_2 \sin(0)$ $\longrightarrow c_1 = 0$

$X(a) = 0 = c_2 \sin(\sqrt{\lambda}a)$

$\sqrt{\lambda}a = n\pi$ if $n = 1, 2, 3, \dots$

$$\boxed{\lambda_n = \frac{n^2\pi^2}{a^2}, \quad X_n = \sin\left(\frac{n\pi x}{a}\right) \quad \text{if } n = 1, 2, 3 \dots}$$

Solve the $Y$ equation:

$$\begin{cases} Y_n'' - \lambda_n Y_n = 0 \\ Y_n(b) = 0 \end{cases}$$

$\lambda_n = \frac{n^2\pi^2}{a^2}$, $Y_n(y)$

Assume $Y_n = e^{ry}$

char. eqn: $r^2 - \lambda_n = 0 = r^2 - \frac{n^2\pi^2}{a^2} = 0$

roots: $r = \pm \frac{n\pi}{a}$

gen. soln: $Y_n = C_1 e^{n\pi y/a} + C_2 e^{-n\pi y/a}$

Recall that:

$\cosh(t) = \frac{e^t + e^{-t}}{2}$ $\qquad \sinh(t) = \frac{e^t - e^{-t}}{2}$

$$Y = c_1 e^{ry} + c_2 e^{-ry}$$

$$= B_1\left(\frac{e^{ry} + e^{-ry}}{2}\right) + B_2\left(\frac{e^{ry} - e^{-ry}}{2}\right)$$

$$= B_1 \cosh(ry) + B_2 \sinh(ry)$$

We also represent our solution as:

$\cosh(t) = \frac{e^{t}+e^{-t}}{2}$ $\quad \sinh(t) = \frac{e^{t}-e^{-t}}{2}$

we also represent our solution as:

$$Y_n = \underset{(B_1)}{C_1} \cosh\left(\frac{n\pi y}{a}\right) + \underset{(B_2)}{C_2} \sinh\left(\frac{n\pi y}{a}\right)$$

$= B_1 \cosh(ry) + B_2 \sinh(ry)$

this is true if

$B_1 + B_2 = C_1$

$B_1 - B_2 = C_2$

can find $B_1$ and $B_2$

NOTE: For the rectangular region $R$, we want to use sinh and cosh instead of $e^{ry}, e^{-ry}$

Impose BC:

$$Y_n(b) = 0 = C_1 \cosh\left(\frac{n\pi b}{a}\right) + C_2 \sinh\left(\frac{n\pi b}{a}\right)$$

solve for $C_2$ in terms of $C_1$

$$C_2 = -\frac{C_1 \cosh\left(\frac{n\pi b}{a}\right)}{\sinh\left(\frac{n\pi b}{a}\right)}$$

so $$Y_n(y) = C_1 \cosh\left(\frac{n\pi y}{a}\right) + \left(\frac{-C_1 \cosh\left(\frac{n\pi b}{a}\right)}{\sinh\left(\frac{n\pi b}{a}\right)}\right) \sinh\left(\frac{n\pi y}{a}\right)$$

pull out a factor of $\frac{C_1}{\sinh\left(\frac{n\pi b}{a}\right)}$

$$= \frac{C_1}{\sinh\left(\frac{n\pi b}{a}\right)} \underbrace{\left[\cosh\left(\frac{n\pi y}{a}\right)\sinh\left(\frac{n\pi b}{a}\right) - \sinh\left(\frac{n\pi y}{a}\right)\cosh\left(\frac{n\pi b}{a}\right)\right]}$$

Hyperbolic Trig Identity:

$$\sinh(\alpha - \beta) = \sinh(\alpha)\cosh(\beta) - \cosh(\alpha)\sinh(\beta)$$

Let $\alpha = \frac{n\pi b}{a}$ $\quad \beta = \frac{n\pi y}{a}$

$$\alpha - \beta = \frac{n\pi b}{a} - \frac{n\pi y}{a} = \frac{n\pi}{a}(b-y)$$

so $$Y_n = \underbrace{\left(\frac{C_1}{\sinh\left(\frac{n\pi b}{a}\right)}\right)}_{\text{just a constant}} \sinh\left(\frac{n\pi}{a}(b-y)\right)$$

call this $c_n$

$$\boxed{Y_n = c_n \sinh\left(\frac{n\pi}{a}(b-y)\right)}$$

So our family of solutions:

$$u_n(x,y) = X_n(x) Y_n(y) \qquad n = 1, 2, 3, \dots$$

By the Principle of Superposition:

$$\boxed{u(x,y) = \sum_{n=1}^{\infty} c_n \sinh\left(\frac{n\pi}{a}(b-y)\right) \sin\left(\frac{n\pi x}{a}\right)}$$

Impose the last BC

$$u(x,0) = f(x)$$

$$u(x,0) = f(x) = \sum_{n=1}^{\infty} c_n \sinh\left(\frac{n\pi}{a}(b-\cancel{0})\right) \sin\left(\frac{n\pi x}{a}\right)$$

$$\sum_{n=1}^{\infty} \left[c_n \sinh\left(\frac{n\pi b}{a}\right)\right] \sin\left(\frac{n\pi x}{a}\right)$$

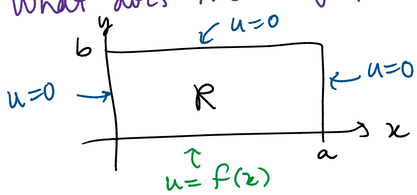
$$u(x,0) = f(x) = \sum_{n=1}^{\infty} C_n \sinh\left(\frac{n\pi}{a}(b-\cancel{0})\right) \sin\left(\frac{n\pi x}{a}\right)$$

$$= \underbrace{\sum_{n=1}^{\infty} \left[C_n \sinh\left(\frac{n\pi b}{a}\right)\right] \sin\left(\frac{n\pi x}{a}\right)}_{\text{almost looks a Fourier Sine Series}}$$

let $C_n \sinh\left(\frac{n\pi b}{a}\right) = b_n = \frac{2}{a}\int_0^a f(x)\sin\left(\frac{n\pi x}{a}\right)dx$

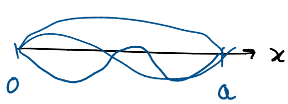
$$\boxed{C_n = \frac{2}{a\sinh\left(\frac{n\pi b}{a}\right)} \int_0^a f(x)\sin\left(\frac{n\pi x}{a}\right)dx}$$

Q: What does mean graphically?




in $x$

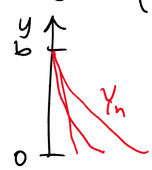
$X_n = \sin\left(\frac{n\pi x}{a}\right)$




$$u = \sum_{n=1}^{\infty} C_n \sinh\left(\frac{n\pi}{a}(b-y)\right)\sin\left(\frac{n\pi x}{a}\right)$$

in $y$

$Y_n = \sinh\left(\frac{n\pi}{a}(b-y)\right)$




$Y_n(b) = \sinh(0) = 0$