Lecture 21

Section 9.7 - Part 2
Laplace's Equation

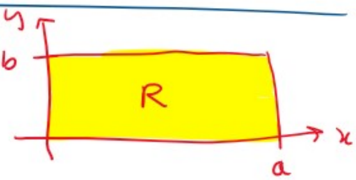
Announcements:
Online HW + A5 due Tues July 20
Midterm 2 on Thurs July 22
→ study guide + practice problems on BrightSpace

Warm up:

Last class, we solved Laplace's Equation

$$u_{xx} + u_{yy} = 0$$

in the rectangle R



using Separation of variables
$$u(x,y) = X(x)\,Y(y)$$
and separation constant $-\lambda$

Q: What are the 2 ODEs you get for X and Y?

A: plug in $u = XY$ into PDE

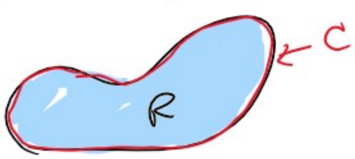
$$X''Y + XY'' = 0$$

$$\frac{X''}{X} = -\frac{Y''}{Y} = -\lambda$$

$$X'' + \lambda X = 0 \qquad\qquad Y'' - \lambda Y = 0$$

Recall, we want Dirichlet Problem:

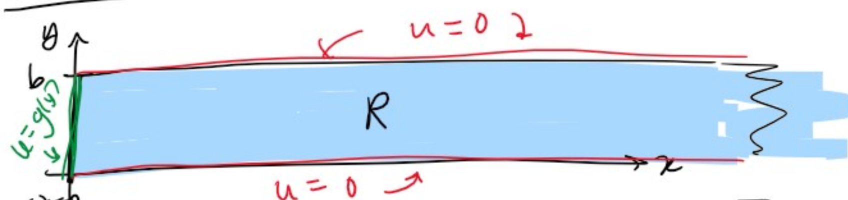
$$\begin{cases} u_{xx} + u_{yy} = 0 & \text{for } x,y \text{ in Region } R \\ u(x,y) = f(x,y) & \text{for } x,y \text{ on Curve } C \end{cases}$$



R can be any region
C is the boundary of R

Last class R - rectangle

I. Semi-Infinite Strip:



$$R = \{(x,y) : x > 0,\ 0 < y < b\}$$

"semi-infinite" because R is infinite in the right direction but has a hard boundary at $x=0$

Equation on R

right direction but has a hard boundary at x=0

GOAL: Solve Laplace Equation on R

$$\begin{cases} u_{xx} + u_{yy} = 0 & x>0, \quad 0<y<b \\ u(x,0) = u(x,b) = 0 \\ u(0,y) = g(y) \\ u(x,y) \text{ is bounded as } x \to +\infty \quad (*) \end{cases}$$

↑ this is a typical BC for unbounded regions

(*) NOTE: $u(x,y)$ is bounded as $x \to +\infty$ can also be written as

" There exists some number $0 \le M < \infty$ such that

$$\lim_{x \to +\infty} |u(x,y)| \le M \text{ "}$$

(so $u(x,y)$ cannot grow to $\pm\infty$)

Solve using separation of variables

$u(x,y) = X(x) Y(y)$

plug into PDE

$u_{xx} + u_{yy} = 0$

$X''Y + XY'' = 0$

$\frac{X''}{X} = \frac{-Y''}{Y} = -\lambda$ (start w/ sep. const $-\lambda$ may change later)

2 ODEs

$X'' + \lambda X = 0$ $\qquad$ $Y'' - \lambda Y = 0$

Let's evaluate the BC

$u(x,0) = 0 = X(x)Y(0)$ → $Y(0) = 0$

$u(x,b) = 0 = X(x)Y(b)$ → $Y(b) = 0$

$u(0,y) = g(y) = X(0)Y(y)$ save for later

$\lim_{x\to\infty} |u(x,y)| \le M$

$\lim_{x\to\infty} |X(x)Y(y)| \le M$ → $\lim_{x\to+\infty} |X(x)| \le M_2$

ODEs:

$x \to \infty$

ODEs:

$$\begin{cases} X'' + \lambda X = 0 \\ \lim_{x \to +\infty} |X(x)| \le M \end{cases} \qquad \star \begin{cases} Y'' - \lambda Y = 0 \\ Y(0) = Y(b) = 0 \end{cases}$$

Q: Should we keep separation constant as $-\lambda$?

A: NO. Rule: put $+\lambda$ on the ODE with 2 BC

So our separation constant should be $+\lambda$

$$\frac{X''}{X} = -\frac{Y''}{Y} = +\lambda$$

$$\begin{cases} X'' - \lambda X = 0 \\ \lim_{x \to \infty} |X(x)| \le M \end{cases} \qquad (*) \begin{cases} Y'' + \lambda Y = 0 \\ Y(0) = Y(b) = 0 \end{cases}$$

Solve the Y equation first

NOTE: this ODE we've solved many times

$$\boxed{Y_n(y) = \sin\left(\frac{n\pi y}{b}\right) \quad \lambda_n = \frac{n^2\pi^2}{b^2} \quad \text{for } n = 1, 2, 3 \ldots}$$

Solve the X equation:

$$\begin{cases} X_n'' - \frac{n^2\pi^2}{b^2} X_n = 0 \\ \lim_{x \to \infty} |X_n(x)| \le M \end{cases}$$

assume $X_n = e^{rx}$

char eqn: $r^2 - \frac{n^2\pi^2}{b^2} = 0$ roots: $r = \pm \frac{n\pi}{b}$

gen. soln: $X_n(x) = C_1 e^{n\pi x/b} + C_2 e^{-n\pi x/b}$

NOTE: For the semi-infinite strip, leave as exponentials

For the rectangle, change to $C_1 \sinh(rx) + C_2 \cosh(rx)$

Impose BC

Impose BC

$$\lim_{x \to +\infty} |X_n(x)| \le M$$

$$\lim_{x \to +\infty} \Big| \cancel{c_1 e^{n\pi x/b}} + c_2 e^{-n\pi x/b} \Big| \underset{\text{want}}{\le} M$$

$c_1 e^{n\pi x/b} \to +\infty$ as $x \to +\infty$, exp. growth

$c_2 e^{-n\pi x/b} \to 0$ as $x \to \infty$, exp. decay

So let's set $c_1 = 0$

Then $\boxed{X_n(x) = e^{-n\pi x/b} \quad \text{for } n = 1, 2, 3, \ldots}$

So we have the family of solutions:

$$u_n(x,y) = X_n(x)\, Y_n(y) \quad \text{for } n = 1, 2, 3, \ldots$$

By the Principle of Superposition

$$\boxed{u(x,y) = \sum_{n=1}^{\infty} b_n e^{-n\pi x/b} \sin\left(\frac{n\pi y}{b}\right)}$$

Impose last BC:

$$u(0,y) = g(y) = \underbrace{\sum_{n=1}^{\infty} b_n \cancel{e^{-0}} \sin\left(\frac{n\pi y}{b}\right)}_{\text{Fourier Sine Series}}$$

($e^{-0} = 1$)

So the b_n are Fourier coefficients

$$\boxed{b_n = \frac{2}{b} \int_0^b g(y) \sin\left(\frac{n\pi y}{b}\right) dy}$$

<u>Ex</u>: Let $g(y) = 100°C$ be constant

let $b = 10$

Dirichlet problem:

$$\begin{cases} u_{xx} + u_{yy} = 0 & 0 < y < 10, \; x > 0 \\ u(x,0) = u(x,10) = 0 \\ u(0,y) = 100 \\ \lim_{x \to +\infty} |u(x,y)| \le M \end{cases}$$

Our solution is:

$$u(x,y) = \sum_{n=1}^{\infty} b_n e^{-n\pi x/10} \sin\left(\frac{n\pi y}{10}\right)$$

$$\quad = \frac{2}{\ } \int^{10} 100 \sin\left(\frac{n\pi y}{\ }\right) dy$$

$$u(x,y) = \sum_{n=1} \ldots$$

where $b_n = \frac{2}{10}\int_0^{10} 100 \sin\left(\frac{n\pi y}{10}\right) dy$

$$= \ldots$$

$$= \begin{cases} \frac{400}{n\pi} & \text{if } n \text{ odd} \\ 0 & \text{if } n \text{ even} \end{cases}$$

full solution

$$\boxed{u(x,y) = \sum_{n \text{ odd}} \frac{400}{n\pi} e^{-n\pi x/10} \sin\left(\frac{n\pi y}{10}\right)}$$

$$u_{xx} + u_{yy} = 0$$
$$u_{xx} = -u_{yy}$$

Compare Solutions

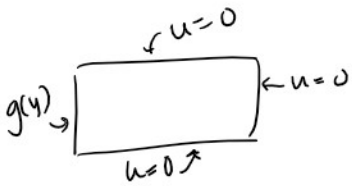
	Rectangle		Semi-Infinite Strip	
BVP	$\begin{cases} u_{xx}+u_{yy}=0 \\ u(0,y)=u(a,y)=0 \\ u(x,0)=f(x) \\ u(x,b)=0 \end{cases}$	$0<x<a$, $0<y<b$; homog in x	$\begin{cases} u_{xx}+u_{yy}=0 \\ u(x,0)=u(x,b)=0 \\ u(0,y)=g(y) \\ \lim_{x\to\infty} \lvert u(x,y)\rvert \le M \end{cases}$	$x>0$, $0<y<b$; homog in y
sep. of vars	$\frac{X''}{X} = -\frac{Y''}{Y} = -\lambda$	not unique to rectangle	$\frac{X''}{X} = -\frac{Y''}{Y} = +\lambda$	not unique to the semi-infinite strip
X eqn	$\begin{cases} X''+\lambda X=0 \\ X(0)=X(a)=0 \end{cases}$ $X_n = \sin\left(\frac{n\pi x}{a}\right)$		$\begin{cases} X''-\lambda X=0 \\ \lim_{x\to+\infty} \lvert X\rvert \le M \end{cases}$ $X_n = e^{-n\pi x/b}$	exp decay
Y eqn	$\begin{cases} Y''-\lambda Y=0 \\ Y(b)=0 \end{cases}$ $Y_n = \sinh\left(\frac{n\pi(b-y)}{a}\right)$		$\begin{cases} Y''+\lambda Y=0 \\ Y(0)=Y(b)=0 \end{cases}$ $Y_n = \sin\left(\frac{n\pi y}{b}\right)$	same problem $a \leftrightarrow b$
u	$u(x,y) = \sum^{\infty} c_n \sin\left(\frac{n\pi x}{a}\right)\sinh\left(\frac{n\pi(b-y)}{a}\right)$		$u(x,y) = \sum^{\infty} b_n e^{-n\pi x/b} \sin\left(\frac{n\pi y}{b}\right)$	

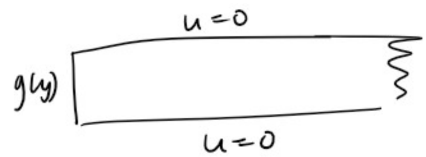
$u(x,0)=u(x,b)=0$ then sep. const $+\lambda$

NOTE: Boundary conditions determine the separation constant $\pm\lambda$

Both the Rectangle + semi-inf. strip can have either $\pm\lambda$

u	$u(x,y) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{a}\right)\sinh\left(\frac{n\pi(b-y)}{a}\right)$	$u(x,y) = \sum_{n=1}^{\infty} b_n e^{-n\pi x/b} \sin\left(\frac{n\pi y}{b}\right)$
coeff	$c_n = \frac{2}{a\sinh\left(\frac{n\pi b}{a}\right)} \int_0^a f(x)\sin\left(\frac{n\pi x}{a}\right)dx$	$b_n = \frac{2}{b}\int_0^b g(y)\sin\left(\frac{n\pi y}{b}\right)dy$ Fourier coefficients





$$\begin{cases} u_{xx} + u_{yy} = 0 \\ u(x,0) = u(x,b) = 0 \\ u(y,a) = 0 \\ u(y,0) = g(y) \end{cases}$$

homog BC in y

$$\begin{cases} u_{xx} + u_{yy} \\ u(x,0) = u(x,b) = 0 \\ \lim_{x\to\infty} |u(x,y)| \le M \\ u(y,0) = g(y) \end{cases}$$

$$\frac{X''}{X} = \frac{-Y}{Y} = +\lambda \qquad\qquad \frac{X''}{X} = \frac{-Y}{Y} = +\lambda$$

$$\begin{cases} Y'' + \lambda Y = 0 \\ Y(0) = Y(b) = 0 \end{cases}$$

$$Y_n = \sin\left(\frac{n\pi y}{b}\right) \qquad n = 1, 2, 3, \ldots$$

$$\begin{cases} X'' - \lambda X = 0 \\ X(a) = 0 \end{cases} \qquad\qquad \begin{cases} X'' - \lambda X = 0 \\ \lim_{x\to\infty} |X(x)| \le M \end{cases}$$

$$X_n = \text{"} \sinh\left(\frac{n\pi(a-x)}{b}\right) \text{"} \qquad\qquad X_n = e^{-n\pi x/b}$$

something like this

★ Next Class

- Solve the Dirichlet Problem when R is a circular disk

- ... problems in

when R is a circle.

— Today: solve problems in semi-infinite strip.