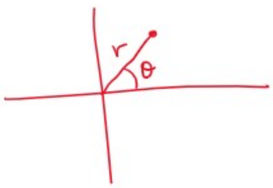
Lecture 23

Sec 9.7 - Part 3

Laplace's Eqn in Polar Coordinates

Announcements:
Online HW + A5 due Tues 7/20
Midterm 2 on Thurs 7/22
Review in class on Wed

Warm up:
Recall polar coordinates in the 2D plane

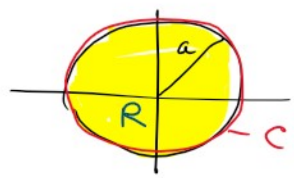


$x = r\cos\theta$
$y = r\sin\theta$

r - radius from origin
θ - angle from the x-axis

I. Laplace's Equation in Polar Coordinates

circular disk



Dirichlet Problem

$$\begin{cases} u_{xx} + u_{yy} = 0 & (x,y) \text{ in } R \\ u(x,y) = f(x,y) & (x,y) \text{ in } C \end{cases}$$

R disk of radius a
C circle w/ radius a

To solve, use polar coordinates (r,θ)

Laplace's Equation in Polar Coordinates

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$$

also write: $u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0$

Then R is $0 \le r < a$
C is the curve $r = a$

Dirichlet Problem in a Circular Disk:

$$\begin{cases} u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0 & 0 \le r < a \\ u(a,\theta) = f(\theta)^{*} & \text{(condition on boundary)} \\ u(r,\theta) = u(r,\theta+2\pi) & (2\pi \text{ periodic in } \theta) \\ u(r,\theta) \text{ is continuous at } r=0 \end{cases}$$

*Assume $f(\theta)$ is 2π periodic

Solve using Sep of Variables
$u(r,\theta) = R(r)\,\Theta(\theta)$
plug int PDE

$$\left(R''\Theta + \frac{1}{r}R'\Theta + \frac{1}{r^2}\Theta'' R = 0\right)\frac{r^2}{R\Theta}$$

$$\frac{r^2R'' + rR'}{R} = -\frac{\Theta''}{\Theta} = \lambda \quad \text{(separation constant)}$$

$$\frac{r^2R'' + rR'}{R} = -\frac{\Theta''}{\Theta} = \lambda \quad \left(\text{separation constant}\right)$$

2 ODEs

choose $+\lambda$ as sep. const

$$\Theta'' + \lambda\Theta = 0$$

$$\frac{r^2R'' + rR'}{R} = \lambda$$

$$r^2R'' + rR' - \lambda R = 0$$

Find the BC

$u(a,\theta) = f(\theta) = R(a)\,\Theta(\theta)$ save for later

$R(r)\Theta(\theta) = u(r,\theta) = u(r,\theta+2\pi) = R(r)\,\Theta(\theta+2\pi)$

$\rightarrow \Theta(\theta) = \Theta(\theta+2\pi) \leftarrow$ is 2π periodic

$u(r,\theta)$ is continuous at $r=0$

$R(r)\Theta(\theta)$ " " $r=0$ $\rightarrow$ $R(r)$ is continuous at $r=0$

$$\begin{cases} r^2R'' + rR - \lambda R = 0 \\ R(r) \text{ continuous @ } r=0 \end{cases} \qquad \begin{cases} \Theta'' + \lambda\Theta = 0 \\ \Theta(\theta) = \Theta(\theta+2\pi) \end{cases}$$

Solve the Θ eqn first

guess $\Theta = e^{r\theta}$

char eqn: $r^2 + \lambda = 0$

roots: $r = \pm i\sqrt{\lambda}$

gen soln: $\Theta = C_1\cos(\sqrt{\lambda}\,\theta) + C_2\sin(\sqrt{\lambda}\,\theta)$

BC: periodicity condition

$$C_1\cos(\sqrt{\lambda}\theta) + C_2\sin(\sqrt{\lambda}\theta) = C_1\cos(\sqrt{\lambda}(\theta+2\pi)) + C_2\sin(\sqrt{\lambda}(\theta+2\pi))$$

phase shift of $2\pi\sqrt{\lambda}$

WANT phase shift to be a multiple of 2π

$\sqrt{\lambda}\,2\pi = n(2\pi)$ for $n=0,1,2,3,\ldots$

$$\boxed{\lambda_n = n^2 \text{ for } n=0,1,2,3,\ldots}$$

$$\boxed{\Theta_n(\theta) = A_n\cos(n\theta) + B_n\sin(n\theta) \quad n=1,2,3,\ldots}$$

Q: What happens if $n=0$?

if $n=0$, $\lambda_0 = 0$

$$\begin{cases} \Theta_0'' = 0 \\ \Theta(\theta) = \Theta_0(\theta+2\pi) \end{cases}$$

$\int \Theta_0'' = \int 0$

$\int \Theta_0' = \int C_1$

$\Theta_0 = C_1\theta + C_2$

BC: (periodicity)

$$\Theta_0(\theta) = C_1\theta + \cancel{C_2} = C_1(\theta+2\pi) + \cancel{C_2}$$

$$0 = 2\pi C_1 \longrightarrow C_1 = 0$$

$$\boxed{\Theta_0(\theta) = C_2 = A_0 \quad n=0}$$

rename constant A_0

Now let's solve the R eqn:

$\lambda_n = n^2$

$$\begin{cases} r^2 R_n'' + r R_n' - n^2 R_n = 0 \\ R_n(r) \text{ is continuous @ } r=0 \end{cases}$$

linear ODE, variable coeff

guess $R_n(r) = r^k$ for some unknown integer k

$$R_n' = k r^{k-1} \qquad R_n'' = k(k-1)r^{k-2}$$

plug into ODE

$$r^2\left[k(k-1)r^{k-2}\right] + r\left[k r^{k-1}\right] - n^2\left[r^k\right] = 0$$

$$\left[k(k-1) + k - n^2\right] r^k = 0$$

$$\underbrace{\left[k^2 - \cancel{k} + \cancel{k} - n^2\right]}_{\text{equal to zero}} r^k = 0$$

$$k^2 - n^2 = 0 \qquad \boxed{k = \pm n}$$

$$R_n(r) = C_n r^n + D_n r^{-n}$$

Impose condition that $R_n(r)$ is continuous @ $r=0$

$$\underbrace{C_n r^n}_{r\to 0} \longrightarrow 0 \quad + \quad \cancel{\underbrace{D_n r^{-n}}_{r\to 0}} \longrightarrow \pm\infty$$

undefined at $r=0$

Let $D_n = 0$

$$\boxed{R_n(r) = C_n r^n \qquad n = 1, 2, 3, \ldots}$$

when $n=0$ solve $r^2 R_0'' + r R_0' = 0$

guess $R_0 = r^k$

$$r^2\left[k(k-1)r^{k-2}\right] + r\left[k r^{k-1}\right] = 0$$

$$\left[k^2 - \cancel{k} + \cancel{k}\right] r^k = 0$$

$$k^2 = 0 \qquad k = 0$$

Sec. 3.3

$u(r) = r$

$= r\, v(r)$

$(r) = \ldots (r)$ solve

$u_1(r) = r$
$u_2(r) = r\,v(r)$
plug in $u_2(r)$ into ODE & solve
$v(r)$
$v(r) = \ln(r)$
$R_0 = C_1 + \cancel{C_2 \ln(r)}$ undefined at $r=0$

$$[k^2 - \cancel{k} + \cancel{k}]\,r^k = 0$$
$$k^2 = 0 \qquad k = 0$$
$$\boxed{R_0(r) = C_0 r^0 = C_0 \quad \text{if } n = 0}$$
constant

✓ this is continuous at $r = 0$

Family of solutions
$$u_n(r,\theta) = r^n\left(a_n \cos(n\theta) + b_n \sin(n\theta)\right)$$
$n = 1, 2, 3, \ldots$

$$u_0(r,\theta) = (C_0)(A_0) \equiv \frac{a_0}{2}$$ rename constant $n=0$

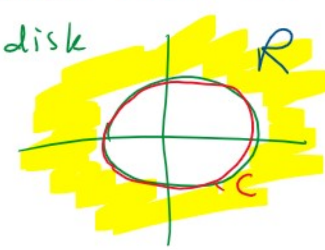
Principle of Superposition:
$$\boxed{u(r,\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos(n\theta) + b_n \sin(n\theta)\right] r^n}$$

Impose last BC:
$$u(a,\theta) = f(\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\underbrace{\left[a_n \cos(n\theta) + b_n \sin(n\theta)\right]}_{\text{Fourier Series}} a^n$$

$$\boxed{\begin{aligned} a_0 &= \frac{1}{\pi}\int_0^{2\pi} f(\theta)\,d\theta \\ a_n &= \frac{1}{\pi a^n}\int_0^{2\pi} f(\theta)\cos(n\theta)\,d\theta \\ b_n &= \frac{1}{\pi a^n}\int_0^{2\pi} f(\theta)\sin(n\theta)\,d\theta \end{aligned}}$$

NOTE: HW problems is to derive $u(r,\theta)$ for the region exterior to the disk

different continuity condition



Example: solution w/ Fourier Cosine Series: (similar to Heat Eqn w/ insulated endpts)

9.7.6: Rectangular Domain

$$\begin{cases} u_{xx} + u_{yy} = 0 & 0 < x < a,\ 0 < y < b \\ u_x(0,y) = u_x(a,y) = 0 & \text{(insulated endpts in } x \text{ dir)} \\ u_y(x,0) = 0 \\ u(x,b) = f(x) \end{cases}$$

Separation of variables

$\left(u(x,b) = f(x)\right.$

Separation of variables

$$\frac{X''}{X} = \frac{-Y''}{Y} = -\lambda$$

(separation const may change this later)

BC:

$u_x(0,y) = 0 = X'(0)Y(y) \rightarrow X'(0) = 0$

$u_x(a,y) = 0 = X'(a)Y(y) \rightarrow X'(a) = 0$

$u_y(x,0) = 0 = X(x)Y'(0) \rightarrow Y'(0) = 0$

save for later

$u(x,b) = f(x)$

$$\begin{cases} X'' + \lambda X = 0 \\ X'(0) = X'(a) = 0 \end{cases} \qquad \begin{cases} Y'' - \lambda Y = 0 \\ Y'(0) = 0 \end{cases}$$

2 BC in X and $+\lambda$

✓ keep sep. const as $-\lambda$

Solve X first

guess $X = e^{rx}$

char. eqn: $r^2 + \lambda = 0$

roots: $r = \pm i\sqrt{\lambda}$

gen soln: $X = c_1 \cos(\sqrt{\lambda}x) + \cancel{c_2 \sin(\sqrt{\lambda}x)}$

$X' = -c_1\sqrt{\lambda}\sin(\sqrt{\lambda}x) + c_2\sqrt{\lambda}\cos(\sqrt{\lambda}x)$

BC: $X'(0) = 0 = -c_1\sqrt{\lambda}\,\cancelto{0}{\sin(0)} + c_2\sqrt{\lambda}\,\cancelto{1}{\cos(0)}$

$c_2 = 0$

$X'(a) = 0 = -c_1\sqrt{\lambda}\sin(\sqrt{\lambda}a)$

$\sqrt{\lambda}\,a = n\pi \qquad n = 1, 2, 3, \ldots$

$$\boxed{\begin{gathered}\lambda_n = \frac{n^2\pi^2}{a^2} \\ X_n = \cos\left(\frac{n\pi x}{a}\right) \quad n = 1, 2, 3, \ldots\end{gathered}}$$

implies soln is a Fourier Cosine Series so there should be an $n=0$ term

if $n=0$ $\lambda_0 = 0$ $\begin{cases} X_0'' = 0 \\ X_0'(0) = X_0'(a) = 0 \end{cases}$

$\int X_0'' = \int 0$ BC: $X_0'(0) = c_1\big|_{x=0} = 0$

$\int X_0' = \int c_1$ $c_1 = 0$

$\int X_0'' = \int 0$ BC: $X_0'(0) = C_1|_{x=0}$

$C_1 = 0$

$\int X_0' = \int C_1$

$X_0 = \cancel{C_1 x} + C_2$ $X_0'(a) = 0 = C_1$

$$\boxed{X_0(x) \equiv 1 \quad \text{if } n=0}$$

Now solve for Y

$$\begin{cases} Y_n'' - \frac{n^2\pi^2}{a^2} Y_n = 0 \\ Y_n'(0) = 0 \end{cases} \rightarrow \boxed{Y_n = \cosh\left(\frac{n\pi y}{a}\right) \quad n = 1, 2, 3, \ldots}$$

check the $n=0$

$$\begin{cases} Y_0'' = 0 \\ Y_0'(0) = 0 \end{cases} \longrightarrow \boxed{Y_0(y) \equiv \underbrace{1}_{\text{constant}} \quad \text{if } n=0}$$

Suppose we had

$$\begin{cases} Y_0'' = 0 \\ Y_0(b) = 0 \end{cases}$$

$Y_0 = C_1 y + C_2$

$Y_0(b) = 0 = C_1 b + C_2$

$C_2 = -C_1 b$

$Y_0(y) = C_1 y - C_1 b$

$$\boxed{Y_0 = C_1(y-b)}$$

$u_0 = X_0 Y_0 = (1) C_1 (y-b)$

$$\boxed{u_0 = C_0(y-b)}$$

Impose BC

☆ $u(x,y) = \left(\frac{a_0}{2}\right)(y-b) + \sum_{n=1}^{\infty} \ldots$

$u(x,0) = f(x) = \left(\frac{a_0}{2}\right)(0-b) + \sum_{n=1}^{\infty} \ldots$

$\frac{b a_0}{2} = \left(\frac{a_0}{2}\right) = \frac{1}{2}\left(\frac{2}{a}\int_0^a f(x)\,dx\right)$

$$\boxed{a_0 = \frac{2}{ab}\int_0^a f(x)\,dx}$$

NOTE: In general $Y_0(y) = Ay + B$ could be linear in y

Family of solutions:

$u_n(x,y) = X_n Y_n = \cos\left(\frac{n\pi x}{a}\right)\cosh\left(\frac{n\pi y}{a}\right) \quad n = 1, 2, 3, \ldots$

$u_0(x,y) = X_0 Y_0 = (1)(1) = 1 \quad$ if $n = 0$

Principle of superposition:

$$\boxed{u(x,y) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{a}\right)\cosh\left(\frac{n\pi y}{a}\right)}$$

Impose the last BC

$$u(x,b) = f(x) = \underbrace{\frac{a_0}{2} + \sum_{n=1}^{\infty} \underbrace{\left[a_n \cosh\left(\frac{n\pi b}{a}\right)\right]}_{\text{constant}} \cos\left(\frac{n\pi x}{a}\right)}_{\text{looks like a Fourier Cosine Series}}$$

$$\boxed{a_0 = \frac{2}{a}\int_0^a f(x)\,dx}$$

$$\left[a_n \cosh\left(\frac{n\pi x}{a}\right)\right] = \text{Fourier Coefficient} = \frac{2}{a}\int_0^a f(x)\cos\left(\frac{n\pi x}{a}\right)dx$$

$$\boxed{a_n = \frac{2}{a\cosh\left(\frac{n\pi x}{a}\right)}\int_0^a f(x)\cos\left(\frac{n\pi x}{a}\right)dx}$$