Lecture 24

☆ Section 10.1 – Part 1

Sturm–Liouville Problems

Announcements:

Online HW + A5 due Today @ 11:59 pm

Office Hours Today @ 2:30–3:30pm

Midterm 2 on Thursday

Review in class on Wednesday

Warm up:

When solving Laplace's Eqn

$u_{xx} + u_{yy} = 0$

using separation of variables, we often derive the end point value problems:

$$(A) \begin{cases} X'' + \lambda X = 0 \\ X(0) = X(L) = 0 \end{cases} \qquad (B) \begin{cases} X'' + \lambda X = 0 \\ X'(0) = X'(L) = 0 \end{cases}$$

Q: What are the solutions to (A) and (B)?

(A)

$$\lambda_n = \frac{n^2\pi^2}{L^2} \quad n = 1, 2, 3, \ldots$$

$$X_n = \sin\left(\frac{n\pi x}{L}\right)$$

(B)

$$\lambda_n = \frac{n^2\pi^2}{L^2} \quad n = 1, 2, 3, \ldots$$

$$X_n = \cos\left(\frac{n\pi x}{L}\right)$$

$n = 0 \quad \lambda_0 = 0$

$X_0 = 1$

I. Sturm–Liouville Problems:

When we solve PDEs using Sep of Vars we often end up with an eigenvalue problem

$$X'' + \lambda X = 0 \qquad \left(\text{can also write } X'' = -\lambda X\right)$$

Suppose we have BC $X(0) = X(L) = 0$

eigenvalues: $\lambda_n = \frac{n^2\pi^2}{L^2} \quad n = 1, 2, 3, \ldots$

eigenfunctions: $X_n(x) = \sin\left(\frac{n\pi x}{L}\right)$

Then we can represent the general solution of the ODE as a linear combination (Principle of Superposition)

$$X(x) = \sum_{n=1}^{\infty} c_n X_n = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{L}\right)$$

This is a special case of a more general

This is a special case of a more general class of problems

<u>Def</u>: A <u>Sturm-Liouville Problem</u> (SLP) is an endpoint value problem of the form:

$$\begin{cases} \frac{d}{dx}\left[p(x)\frac{dy}{dx}\right] - q(x)y + \lambda r(x) y = 0 & (a<x<b) \\ \alpha_1 y(a) - \alpha_2 y'(a) = 0 \\ \beta_1 y(b) + \beta_2 y'(b) = 0 \end{cases}$$

α_1, α_2 constant, but cannot both be zero

β_1, β_2 const, but cannot both be zero

Here λ is the eigenvalue

<u>GOAL</u>: Find λ (eigenvalue) and $y(x)$ (eigenfunction)

<u>Ex</u>: (A) $\begin{cases} y'' + \lambda y = 0 \\ y(0) = 0 \\ y(1) = 0 \end{cases}$ $\quad \overset{a}{0} < x < \overset{b}{1}$

Here
$p(x) = 1$, $q(x) \equiv 0$
$r(x) = 1$
$\alpha_1 = 1$, $\alpha_2 = 0$
$\beta_1 = 1$, $\beta_2 = 0$

$$\frac{d}{dx}\left[p(x)y'\right] - q(x)y + \lambda r(x) y = y'' + \lambda y = 0$$

$$\cancel{p'(x)}y' + p(x)y'' - q(x)y + \lambda r(x) y = y'' + \lambda y + 0$$

$p'(x) = 0$ $\quad p(x) = 1$ $\quad r(x) = 1$ $\quad q(x) = 0$

$$\alpha_1 y(0) - \alpha_2 y'(0) = 0 = y(0) + 0\,y'(0)$$

$\alpha_1 = 1$ $\quad \alpha_2 = 0$

$$\beta_1 y(1) + \beta_2 y'(1) = 0 = y(1) + 0\,y'(1)$$

$\beta_1 = 1$ $\quad \beta_2 = 0$

<u>NOTE</u>: Sturm-Liouville problems always have the trivial solution $y(x) \equiv 0$

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GOAL: Find all the eigenvalues that result in nontrivial solutions $(y_n(x) \not\equiv 0)$

Thm 1 (Sturm-Liouville Eigenvalues)

Suppose $p(x), p'(x), q(x), r(x)$ are all continuous on $[a,b]$ and that

$p(x) > 0$ and $r(x) > 0$ at each point in $[a,b]$

Then, the eigenvalues form an increasing sequence of real numbers

$$\lambda_1 < \lambda_2 < \lambda_3 < \dots < \lambda_n < \dots$$

with $\lim_{n\to\infty} \lambda_n = +\infty$

Furthermore, if $q(x) \geq 0$ and $\alpha_1, \alpha_2, \beta_1, \beta_2 \geq 0$ then all λ's are also nonnegative $(\lambda_n \geq 0)$

Def: A Sturm-Liouville problem is called regular if it satisfies the conditions of Thm 1.

Ex: $\begin{cases} y'' + \lambda y = 0 & 0 < x < L \\ y(0) = 0 & \\ h\,y(L) + y'(L) = 0 & (h > 0) \end{cases}$

Here: ODE same as before, $p(x) = 1$, $q(x) = 0$, $r(x) = 1$

$$\alpha_1 y(0) - \alpha_2 y'(0) = 0 = y(0)$$

$\alpha_1 = 1 \qquad \alpha_2 = 0$

$$\beta_1 y(L) + \beta_2 y'(L) = 0 = h\,y(L) + y'(L)$$

$\beta_1 = h > 0 \qquad \beta_2 = 1$

$\beta_1 = h > 0 \quad \beta_2$

So these satisfy conditions of Thm 1
so this SLP is <u>regular</u>

Thm 1 $\rightarrow 0 \le \lambda_1 < \lambda_2 < \lambda_3 < \dots < \lambda_n < \dots$
and $\lambda_n \ge 0$

<u>GOAL</u>: Find λ_n and $y_n(x)$

By Thm 1 we know there are no $\lambda < 0$

1. Find the values of λ_n
2 cases: (i) $\lambda = 0$
(ii) $\lambda = +\alpha^2 > 0$ where $\alpha > 0$

(i) $\lambda = 0$
$$\begin{cases} y'' = 0 & (0 < x < L) \\ y(0) = 0 \\ h\,y(L) + y'(L) = 0 \end{cases}$$

Integrate both sides
$$\int y''\,dx = \int 0\,dx$$
$$\int y' = \int A\,dx$$
$$y(x) = Ax + \cancel{B}$$

Impose BC
$y(0) = 0 = A \cdot \cancel{0} + B \longrightarrow B = 0$

$h\,y(L) + y'(L) = 0 = h[AL] + [A]$

$0 = A[\underbrace{hL+1}_{>0}]$

assume $h > 0$, $L > 0$

So $A = 0$

$y(x) \equiv 0$ trivial solution

No nontrivial solutions when $\lambda = 0$

(ii) $\lambda = +\alpha^2 > 0$
$$\begin{cases} y'' + \alpha^2 y = 0 & 0 < x < L \\ y(0) = 0 \end{cases}$$

$$\begin{cases} y'' + \alpha^2 y = 0 \\ y(0) = 0 \\ h y(L) + y'(L) = 0 \end{cases}$$

guess $y = e^{rx}$

char eqn: $r^2 + \alpha^2 = 0$

roots: $r = \pm i\alpha$

gen soln: $y(x) = A\cancel{\cos}(\alpha x) + B\sin(\alpha x)$

Impose BC:

$$y(0) = 0 = A\cos(0)^{\nearrow 1} + B\cancel{\sin(0)}^{\nearrow 0} \Rightarrow A = 0$$

$$hy(L) + y'(L) = 0 = h\left[B\sin(\alpha L)\right] + \left[B\alpha\cos(\alpha L)\right]$$

$$\frac{\cancel{hB}\sin(\alpha L)}{\cancel{Bh}\cos(\alpha L)} = -\frac{\cancel{B}\alpha\cancel{\cos(\alpha L)}}{\cancel{B}h\cancel{\cos(\alpha L)}}$$

$$\tan(\alpha L) = -\frac{\alpha}{h}\frac{(L)}{(L)} = -\frac{(\alpha L)}{hL}$$

$$\boxed{\tan(\alpha L) = -\frac{(\alpha L)}{hL}}$$

GOAL: Find α to get $\lambda = +\alpha^2$

But this an implicit function, can't solve it explicitly. Two options

(1) solve numerically using computer

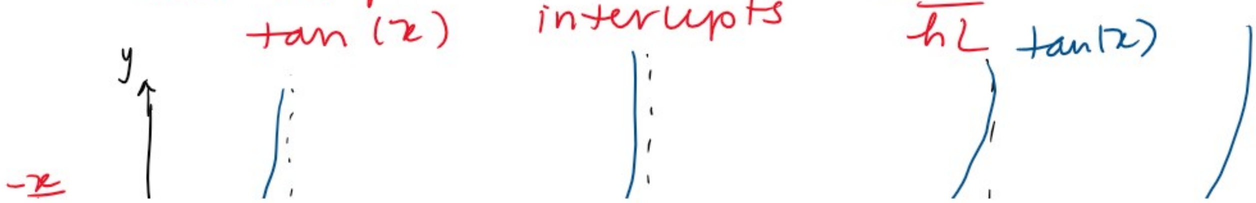
(2) graphically visualize the solns

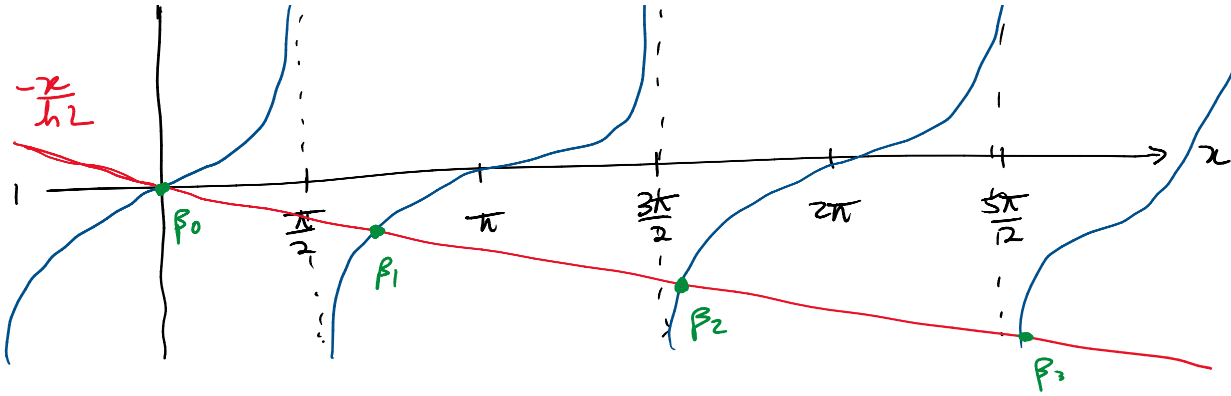
option 2: Visualize solutions

rename $x = \alpha L$

solve for x $\quad \boxed{\tan(x) = \frac{-x}{hL}}$

find the points x where $\tan(x)$ interupts $-\frac{x}{hL}$





Want $x > 0$ because $\alpha > 0$ and $x = \alpha L$

$\beta_0, \beta_1, \beta_2, \ldots$ are the interceptions of $\tan(x)$ and $-\frac{x}{hL}$

so $\beta_n = x = \alpha L$ $\qquad \alpha_n = \frac{\beta_n}{L}$

and $\lambda_n = +\alpha_n^2 = \left(\frac{\beta_n}{L}\right)^2$

Observe that β_n satisfy $0 < \beta_1 < \beta_2 < \beta_3 \ldots$ so is $0 < \lambda_1 < \lambda_2 < \ldots$

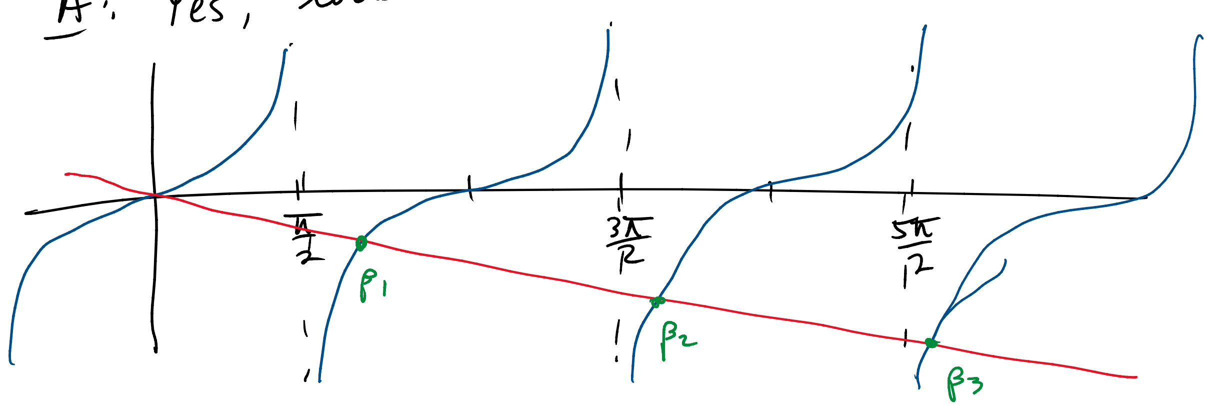
$$\boxed{\text{eigenvalues} \quad \lambda_n = \frac{\beta_n^2}{L^2} \quad n = 1, 2, 3, \ldots}$$

We derived that eigenfunctions $y_n = \sin(\alpha x)$

$$\boxed{\text{eigenfunctions} \quad y_n(x) = \sin\left(\frac{\beta_n x}{L}\right)}$$

Q: Can we estimate the value of β_n for n large?

A: Yes, look at sketch



$\beta_1 = \frac{\pi}{2} + \underbrace{\Delta_1}_{\text{a little bit}}$

$\beta_2 = \frac{3\pi}{2} + \Delta_2$

$\beta_3 = \frac{5\pi}{2} + \Delta_3$

so, we can represent β_n as

$$\beta_n = (2n-1)\frac{\pi}{2} + \Delta_n$$

Take the limit as $n \to \infty$

$$\lim_{n\to\infty} \beta_n = \lim_{n\to\infty} \underbrace{(2n-1)\frac{\pi}{2}}_{\to +\infty} + \cancel{\Delta_n} \to 0$$

graphically, this "little bit" decreases as n gets large

$$= +\infty$$

so when n is large, estimate that

$$\boxed{\beta_n \approx (2n-1)\frac{\pi}{2}}$$

Friday: → properties of eigenfunctions
→ eigenfunction expansions

$y'' + \lambda y = 0$

$y(0) = y(L) = 0$

$r^2 + \lambda = 0$

$r = \pm i\sqrt{\lambda}$

$y = A\cos(\sqrt{\lambda}x) + B\cos(\sqrt{\lambda}x)$

$y_n = \sin\left(\frac{n\pi x}{L}\right)$

$y(0) = 0 \longrightarrow A = 0$

$y(L) = 0 \longrightarrow L\sqrt{\lambda} = n\pi \qquad \lambda = \frac{n^2\pi^2}{L^2}$

$y(l)=0$ $\longrightarrow$

$$y = \cancel{A\cos\left(\frac{n\pi x}{l}\right)} + B\sin\left(\frac{n\pi x}{l}\right)$$

choose $B=1$

$$\begin{cases} y'' + \lambda y = 0 \\ y(0) = 0 \end{cases} \qquad y = B\sin\left(\frac{n\pi x}{l}\right)$$

$$u_n = X_n(x)\,Y_n(y)$$

Principle of S.

$$u(x,y) = \sum c_n u_n = \sum_{n=1}^{\infty} c_n X_n Y_n$$