Lecture 25

Section 10.1 - Part 2

Eigenfunction Expansions

Announcements:
- Online HW + A6 due Tues 7/27

Warm up:

Recall, a Sturm-Liouville Problem (SLP) is defined as

$$\begin{cases} \frac{d}{dx}\left[p(x)\frac{dy}{dx}\right] - q(x)y + \lambda r(x) y = 0 & (a < x < b) \\ \alpha_1 y(a) - \alpha_2 y'(a) = 0 \\ \beta_1 y(b) + \beta_2 y'(b) = 0 \end{cases}$$

where $\alpha_1, \alpha_2, \beta_1, \beta_2 > 0$ are all constants

GOAL: Find λ and $y(x)$

I. Eigenvalues & Eigenfunctions:

Def: A SLP is called regular if
$p(x), p'(x), q(x), r(x)$ are continuous
$p(x), r(x), q(x), \alpha_1, \alpha_2, \beta_1, \beta_2 > 0$
(conditions of Thm 1)

Last class:

Thm 1: If SLP is regular, then it has eigenvalues

$$0 \le \lambda_1 < \lambda_2 < \cdots < \lambda_n < \cdots$$

with $\lim_{n\to\infty} \lambda_n = +\infty$

and each λ_n has a corresponding eigenfunction $y_n(x)$

Thm 2: (Orthogonality of eigenfunctions)
Let SLP be regular, Let $y_i(x)$ and $y_j(x)$ be eigenfunctions associated with λ_i and λ_j
$(\lambda_i \neq \lambda_j)$

Then $$\int_a^b y_i(x)\, y_j(x)\, \underbrace{r(x)}_{\text{weight function}}\, dx = 0$$

Ex: SLP: $\begin{cases} y'' + \lambda y = 0 & 0 < x < \pi \\ y(0) = y(\pi) = 0 \end{cases}$

Here $p(x) = 1$, $q(x) \equiv 0$, $r(x) = 1$
$\alpha_1 = \beta_1 = 1$ $\qquad \alpha_2 = \beta_2 = 0$

eigenvalues: $\lambda_n = n^2$
eigenfunctions: $y_n(x) = \sin(nx)$

orthogonality: assume $\lambda_i \neq \lambda_j$
$y_i(x) = \sin(ix)$ $\qquad y_j(x) = \sin(jx)$ $\qquad r(x) = 1$

WTS: $$\int_0^\pi \sin(ix)\sin(jx)\,(1)\,dx = 0$$

showed this in Sec 9.1

II. Eigenfunction Expansions:

If $y_n(x)$ are eigenfunctions of a regular ... write any function $f(x)$

If $y_n(x)$ are eigenfun[illegible] SLP, then we can write any function $f(x)$ as a linear combination of $y_n(x)$

$$f(x) = \sum_{n=1}^{\infty} c_n y_n(x)$$

where the c_n are constant coeff.

Ex: Fourier series are an example

$$y_n(x) = \sin(nx)$$

$$f(x) = \sum_{n=1}^{\infty} c_n \sin(nx) \longleftarrow \text{Fourier Sine Series}$$

coefficients c_n are Fourier Coeff

$$c_n = b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin(nx)\,dx$$

Q: How to compute c_n if the $y_n(x)$ are NOT Fourier series functions?

Hint: use orthogonality

$$f(x) = \sum_{n=1}^{\infty} c_n y_n(x)$$

multiply both sides by $y_m(x) r(x)$

$$f(x) y_m(x) r(x) = \sum_{n=1}^{\infty} c_n y_n(x) y_m(x) r(x)$$

integrate over $[a, b]$

$$\int_a^b f(x) y_m(x) r(x)\,dx = \sum_{n=1}^{\infty} c_n \int_a^b y_n(x) y_m(x) r(x)\,dx$$

$$\int_a f(x)\,y_m(x)\,r(x)\,dx = \sum_{n=1} c_n \underbrace{\int_a y_n \cdots y_m \cdots}_{\text{use orthogonality}}$$

$$\begin{cases} 0 & \text{if } n \neq m \\ ? & \text{if } n = m \end{cases}$$

$$= c_m \int_a^b \left[y_m(x)\right]^2 r(x)\,dx$$

$$\boxed{c_m = \frac{\int_a^b f(x)\,y_m(x)\,r(x)\,dx}{\int_a^b \left[y_m(x)\right]^2 r(x)\,dx}}$$

Ex: (same SCP from Lecture 24)

$$\begin{cases} y'' + \lambda y = 0 & 0 < x < L \\ y(0) = 0 & \\ h\,y(L) + y'(L) = 0 & h > 0 \end{cases}$$

Last class we found eigenvalues $\lambda_n = \dfrac{\beta_n^2}{L^2}$

eigenfunctions $y_n = \sin\left(\dfrac{\beta_n x}{L}\right)$

where β_n are positive solutions $\tan(x) = \dfrac{-x}{hL}$

Q: Expand $f(x) \equiv 1$ using these eigenfunctions

$$1 = f(x) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{\beta_n x}{L}\right)$$

calculate c_n $\qquad (r(x) \equiv 1)$

$$c_n = \frac{\int_0^L (1) \sin\left(\frac{\beta_n x}{L}\right) (1)\, dx}{\int_0^L \sin^2\left(\frac{\beta_n x}{L}\right) (1)\, dx} = \frac{\text{top}}{\text{bottom}}$$

$$\text{top} = \int_0^L \sin\left(\frac{\beta_n x}{L}\right) dx = \left[\frac{-\cos\left(\frac{\beta_n x}{L}\right)}{\frac{\beta_n}{L}} \right]_0^L$$

$$= \frac{L}{\beta_n}\left[-\cos\left(\beta_n \cancel{\tfrac{L}{L}}\right) + \cos(0) \right] = \frac{L}{\beta_n}\left[1 - \cos(\beta_n) \right]$$

Double angle formula

$\cos(2\theta) = 1 - 2\sin^2(\theta)$

$\sin^2(\theta) = \frac{1}{2}\left[1 - \cos(2\theta)\right]$

$\theta = \frac{\beta_n x}{L}$

$$\text{bottom} = \int_0^L \sin^2\left(\frac{\beta_n x}{L}\right) dx$$

$$= \frac{1}{2}\int_0^L 1 - \cos\left(\frac{2\beta_n x}{L}\right) dx$$

$$= \frac{1}{2}\left[x - \frac{\sin\left(\frac{2\beta_n x}{L}\right)}{\frac{2\beta_n}{L}} \right]_0^L$$

$$= \frac{1}{2}\left[L - 0 + \frac{L}{2\beta_n}\left(-\sin\left(2\beta_n \cancel{\tfrac{L}{L}}\right) + \cancel{\sin(0)}^{\,0} \right) \right]$$

double angle formula

$\sin(2\theta) = 2\sin\theta\cos\theta$

$\theta = \beta_n$

$$= \frac{1}{2}\left[L - \frac{L\sin(2\beta_n)}{2\beta_n} \right]$$

$$= \frac{1}{2}\left[L - \frac{\cancel{2} L \sin(\beta_n)\cos(\beta_n)}{\cancel{2}\beta_n} \right]$$

$$= \frac{L}{2\beta_n}\left[\beta_n - \sin(\beta_n)\cos(\beta_n) \right]$$

$$= \frac{L}{2\beta_n}\left[\beta_n - \sin(\beta_n)\cos(\beta_n)\right]$$

$$C_n = \frac{\text{top}}{\text{bottom}} = \frac{\frac{\cancel{L}}{\cancel{\beta_n}}\left[1-\cos(\beta_n)\right]}{\frac{\cancel{L}}{2\cancel{\beta_n}}\left[\beta_n - \sin(\beta_n)\cos(\beta_n)\right]}$$

$$\boxed{C_n = \frac{2\left[1-\cos(\beta_n)\right]}{\beta_n - \sin(\beta_n)\cos(\beta_n)}}$$

$$\boxed{1 = f(x) = \sum_{n=1}^{\infty}\left[\frac{2\left[1-\cos(\beta_n)\right]}{\beta_n - \sin(\beta_n)\cos(\beta_n)}\right]\sin\left(\frac{\beta_n x}{L}\right)}$$

Eigenfunction Expansion

Thm 3: (Convergence of Eigenfunction Series)

Let $y_1, y_2, y_3, \ldots$ be the eigenfunctions of a regular SLP on $[a,b]$. If $f(x)$ is piecewise smooth on $[a,b]$ then the eigenfunction series converges for $a < x < b$ to:

(i) the value $f(x)$ where f is continuous

(ii) the average value $\frac{1}{2}\left[f(x^+) + f(x^-)\right]$ at each point of discontinuity.

☆ **Summary:**

Every regular Sturm-Liouville Problem has the following properties has the following properties

1. An infinite sequence of eigenvalues diverging to $+\infty$ (Thm 1)

$$0 \le \lambda_1 < \lambda_2 < \cdots < \lambda_n < \cdots$$

$$\lim_{n \to \infty} \lambda_n = +\infty$$

2. The eigenfunctions $y_n(x)$ are orthogonal with weight fcn $r(x)$ (Thm 2)

$$\int_a^b y_n(x)\, y_m(x)\, r(x)\, dx = 0 \qquad \text{if } \lambda_n \neq \lambda_m$$

3. Any piecewise smooth function $f(x)$ can be represented by an eigenfcn series (Thm 3)

$$f(x) = \sum_{n=1}^{\infty} c_n y_n(x)$$

$$c_n = \frac{\int_a^b f(x)\, y_n(x)\, r(x)\, dx}{\int_a^b [y_n(x)]^2\, r(x)\, dx}$$