Lecture 27

Section 7.1: Laplace Transforms

Announcements:
Online HW + A6 due Tues 7/27
Office Hours Today @ 2:30 - 3:30pm

Warm up:
Use Euler's formula: $e^{it} = \cos(t) + i\sin(t)$
To write $\cos(t)$ in terms of complex exponentials.

Ans:

$$\begin{aligned} e^{it} &= \cos(t) + i\sin(t) \\ + \; e^{-it} &= \cos(t) - i\sin(t) \\ \hline e^{it} + e^{-it} &= 2\cos(t) + 0 \end{aligned}$$

$$\cos(t) = \frac{1}{2}\left[e^{it} + e^{-it}\right]$$

I. Laplace Transforms:

→ powerful tool to solve linear ODEs with variable coeffs

$$a(t)\,y'' + b(t)\,y' + c(t)\,y = f(t)$$

Def: The Laplace Transform of a function $f(t)$ is defined

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty \underbrace{e^{-st}}_{\text{"kernel" of the transform}} f(t)\,dt$$

Linear ODE ⟶ solutions are exponentials
so the kernel e^{-st} is a good choice

$$f(t) \longmapsto \mathcal{L} \longrightarrow F(s)$$

Ex: Find the Laplace Transform (LT) of $f(t) \equiv 1$

$$\mathcal{L}\{1\} = \int_0^\infty e^{-st}(1)\,dt = \int_0^\infty e^{-st}\,dt$$

indefinite Integral

$$= \lim_{b\to\infty} \int_0^b e^{-st}\,dt = \lim_{b\to\infty}\left[\frac{e^{-st}}{-s}\right]_0^b$$

$$= \lim_{b\to\infty}\left[\underbrace{\frac{e^{-sb}}{-s}}_{\xrightarrow{\text{if } s>0} 0} - \frac{1}{-s}\right] = \frac{1}{s} \quad \text{if } s>0$$

$$\boxed{\mathcal{L}\{1\} = \frac{1}{s} \quad \text{if } s>0}$$

Q: Find the L.T. of $f(t) \equiv 3$

(c) $\mathcal{L}\{3\} = \frac{1}{s^3}$

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(a) $\mathcal{L}\{3\} = \frac{1}{s}$ $\quad$ $\boxed{(b)\ \mathcal{L}\{3\} = \frac{3}{s}}$ $\quad$ (c) $\mathcal{L}\{3\} = \frac{1}{s^3}$

$$\mathcal{L}\{3\} = \int_0^\infty e^{-st}(3)\,dt = 3\underbrace{\int_0^\infty e^{-st}\,dt}_{\mathcal{L}\{1\}} = 3\mathcal{L}\{1\} = \frac{3}{s}$$

$$\boxed{\mathcal{L}\{a f(t)\} = a\mathcal{L}\{f(t)\} \quad \text{where } a \text{ is a constant}}$$

Ex: Find the L.T. of $f(t) = e^{at}$ where a is a const.

$$\mathcal{L}\{e^{at}\} = \int_0^\infty e^{-st}(e^{at})\,dt = \lim_{b\to\infty}\int_0^b e^{-(s-a)t}\,dt$$

$$= \lim_{b\to\infty}\left[\frac{e^{-(s-a)t}}{-(s-a)}\right]_0^b = \lim_{b\to\infty}\left(\underbrace{\frac{e^{-(s-a)b}}{-(s-a)}}_{\to 0} - \frac{1}{-(s-a)}\right)$$

$s - a > 0$

$$\boxed{\mathcal{L}\{e^{at}\} = \frac{1}{s-a} \quad \text{if } s > a}$$

II. Linearity:

Ex: Find the L.T. of $f(t) = 1 + e^{-3t}$

$$\mathcal{L}\{1 + e^{-3t}\} = \int_0^\infty e^{-st}(1 + e^{-3t})\,dt$$

because integral is linear

$$= \int_0^\infty e^{-st}\,dt + \int_0^\infty e^{-st}e^{-3t}\,dt$$

$$= \mathcal{L}\{1\} + \mathcal{L}\{e^{-3t}\}$$

$$\boxed{\mathcal{L}\{1 + e^{-3t}\} = \frac{1}{s} + \frac{1}{s+3} \quad \text{if } s > 0}$$

So the L.T. satisfies

$$\begin{cases} \mathcal{L}\{a f(t)\} = a\mathcal{L}\{f(t)\} \\ \mathcal{L}\{f(t) + g(t)\} = \mathcal{L}\{f(t)\} + \mathcal{L}\{g(t)\} \end{cases}$$

We say the Laplace Transform is <u>linear</u>

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NOTE: This is <u>NOT</u> true for multiplication of fcns

$$\mathcal{L}\{f(t)g(t)\} \cancel{=} \mathcal{L}\{f(t)\}\mathcal{L}\{g(t)\}$$

III. More Transforms

<u>Ex</u>: Find the L.T. of $f(t) = \cos(t)$

$$\mathcal{L}\{\cos(t)\} = \int_0^\infty e^{-st}\cos(t)\,dt \quad \leftarrow \text{difficult integral}$$

instead, expand $\cos(t)$ using Euler's formula

$$\cos(t) = \frac{1}{2}\left[e^{it} + e^{-it}\right]$$

$$\mathcal{L}\{\cos(t)\} = \mathcal{L}\left\{\frac{1}{2}\left[e^{it} + e^{-it}\right]\right\} \leftarrow \text{linearity}$$

$$= \frac{1}{2}\mathcal{L}\{e^{it}\} + \frac{1}{2}\mathcal{L}\{e^{-it}\}$$

$$= \frac{1}{2}\left(\frac{1}{s-i}\right) + \frac{1}{2}\left(\frac{1}{s+i}\right)$$

$s > Re(i) = 0 \qquad s > Re(i) = 0$

find a common denominator $\longrightarrow$ real-valued

$$= \frac{1}{2}\left[\left(\frac{1}{s-i}\right)\frac{(s+i)}{(s+i)} + \left(\frac{1}{s+i}\right)\frac{(s-i)}{(s-i)}\right]$$

$$= \frac{1}{2}\left[\frac{s+\cancel{i}+s-\cancel{i}}{s^2+1}\right] = \frac{1}{\cancel{2}}\left(\frac{\cancel{2}s}{s^2+1}\right)$$

$$\boxed{\mathcal{L}\{\cos(t)\} = \frac{s}{s^2+1} \qquad s > 0}$$

<u>Ex</u>: Find the L.T. of $f(t) = 5t$

$$\mathcal{L}\{5t\} = \int_0^\infty e^{-st}(5t)\,dt = 5\int_0^\infty te^{-st}\,dt$$

Integration by Parts: $u = t \quad dv = e^{-st}dt$

$du = dt \quad v = \frac{e^{-st}}{-s}$

$$= 5\lim\left[\left(\frac{te^{-st}}{\ }\right)^b - \int^b \frac{e^{-st}}{-s}\,dt\right]$$

$$= 5 \lim_{b \to \infty} \left[\left(\frac{te^{-st}}{-s} \right)_0^b - \int_0^b \frac{e^{-st}}{-s} dt \right]$$

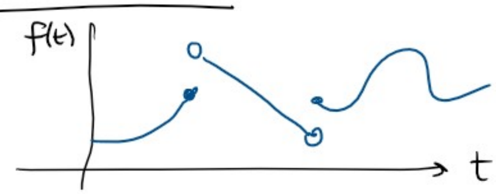
$$= 5 \lim_{b \to \infty} \left[\frac{te^{-st}}{-s} - \frac{e^{-st}}{(-s)^2} \right]_0^b$$

$$= 5 \lim_{b \to \infty} \left[\underbrace{\frac{be^{-sb}}{-s}}_{\xrightarrow[s>0]{} 0} - \underbrace{\frac{e^{-sb}}{s^2}}_{\xrightarrow[s>0]{} 0} - 0 + \frac{1}{s^2} \right]$$

$$= 5 \left[\frac{1}{s^2} \right] \qquad \boxed{\mathcal{L}\{5t\} = \frac{5}{s^2} \quad \text{if } s > 0}$$

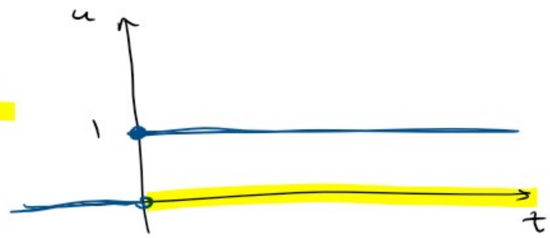
IV. Piecewise Continuous Functions:

function $f(t)$ has finitely many jumps



Ex: $u(t) = \begin{cases} 0 & \text{if } t < 0 \\ 1 & \text{if } t \geq 0 \end{cases}$

unit step function

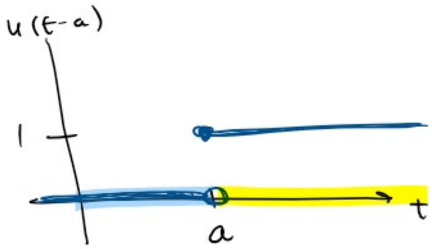


$$\mathcal{L}\{u(t)\} = \int_0^\infty e^{-st} u(t)\, dt = \int_0^\infty e^{-st} (1)\, dt = \mathcal{L}\{1\} = \frac{1}{s}$$

because $u(t) = 1$ if $t \geq 0$

$$\boxed{\mathcal{L}\{u(t)\} = \frac{1}{s} \quad \text{if } s > 0}$$

Ex: $u_a(t) = u(t-a) = \begin{cases} 0 & t < a \\ 1 & t \geq a \end{cases}$



$$\mathcal{L}\{u(t-a)\} = \int_0^\infty e^{-st} u(t-a)\, dt$$

$$= \int_0^a \cancel{e^{-st}(0)}\, dt + \int_a^\infty e^{-st}(1)\, dt$$

$$= \lim_{b\to\infty} \int_a^b e^{-st}\,dt = \lim_{b\to\infty} \left[\frac{e^{-st}}{-s} \right]_a^b$$

$$= \lim_{b\to\infty} \left(\underbrace{\frac{e^{-sb}}{-s}}_{\to 0 \text{ if } s>0} - \frac{e^{-sa}}{-s} \right) = \frac{e^{-as}}{s}$$

$$\boxed{\mathcal{L}\{u(t-a)\} = \frac{e^{-as}}{s} \quad \text{if } s>0}$$

V. Inverse Laplace Transforms:

If $F(s) = \mathcal{L}\{f(t)\}$, then we call $f(t)$ the <u>inverse Laplace transform</u> of $F(s)$

$$\begin{tikzcd} f(t) \arrow[r, "\mathcal{L}"] & F(s) \arrow[l, bend left, "\mathcal{L}^{-1}"] \end{tikzcd}$$

<u>Ex</u>: $\mathcal{L}^{-1}\left\{\frac{1}{s+3}\right\} = ? = e^{-3t}$

$\mathcal{L}\{\underbrace{e^{at}}_{f(t)}\} = \frac{1}{s-a}$ set $a=-3$

$$\mathcal{L}^{-1}\left\{\frac{5}{s^2}\right\} = 5t$$

Before, we showed $\mathcal{L}\{5t\} = \frac{5}{s^2}$

In practice, use a <u>Table of Laplace Transforms</u>

$f(t)$	$F(s)$	
1	$\frac{1}{s}$	$s>0$

1	$\frac{1}{s}$	$s>0$
t	$\frac{1}{s^2}$	$s>0$
t^n ($n \geq 0$ integer)	$\frac{n!}{s^{n+1}}$	$s>0$
e^{at}	$\frac{1}{s-a}$	$s>a$
$\cos(kt)$	$\frac{s}{s^2+k^2}$	$s>0$
$\sin(kt)$	$\frac{k}{s^2+k^2}$	$s>0$
$u(t-a)$	$\frac{e^{-as}}{s}$	$s>0$, $a>0$

Ex: Find the inverse L.T. of $F(s) = \frac{1}{s} - \frac{2}{s-5}$

linearity

$$\mathcal{L}^{-1}\left\{\frac{1}{s} - \frac{2}{s-5}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - 2\mathcal{L}^{-1}\left\{\frac{1}{s-5}\right\}$$

$$= 1 - 2\left(e^{5t}\right) = \boxed{1 - 2e^{5t} = f(t)}$$

Ex: Find $\mathcal{L}\left\{3e^{2t} + 2\sin^2(3t)\right\}$

use trig identity: $2\sin^2\theta = 1 - \cos(2\theta)$

Let $\theta = 3t$

$$= \mathcal{L}\left\{3e^{2t} + 1 - \cos(6t)\right\}$$

linearity

$$= 3\mathcal{L}\left\{e^{2t}\right\} + \mathcal{L}\left\{1\right\} - \mathcal{L}\left\{\cos(6t)\right\}$$

look up in table

$$= 3\left(\frac{1}{s-2}\right) + \left(\frac{1}{s}\right) - \left(\frac{s}{s^2+36}\right)$$

Find a common denominator

$$= \frac{3s^3 + 144s - 72}{s(s-2)(s^2+36)} = F(s)$$

☆ Summary:

- $\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$
- The Laplace Transform is linear
  $\mathcal{L}\{a f(t) + b g(t)\} = a\mathcal{L}\{f(t)\} + b\,\mathcal{L}\{g(t)\}$
- You can take the L.T. of a piecewise continuous function, like the unit step function $u(t-a) = \begin{cases} 0 & t < a \\ 1 & t \geq a \end{cases}$
- To find $\mathcal{L}^{-1}\{F(s)\}$ (inverse L.T.) use a Table of Laplace Transforms.