Lecture 27

Section 7.2: Transformation of Initial Value Problems

Announcements:
Online HW + A6 due Today @ 11:59pm
Office Hours Today @ 2:30-3:30pm

Warm up:
Write down the definition of the Laplace Transform of $f(t)$:

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$$

I. IVPs:

$$\begin{cases} ax'' + bx' + cx = f(t) \\ x(0) = x_0, \quad x'(0) = x_0' \end{cases}$$

GOAL: solve using Laplace Transforms

Procedure:

1. Take the L.T. of both sides
$$\mathcal{L}\{ax'' + bx' + cx\} = \mathcal{L}\{f(t)\}$$

2. Use linearity
$$a\mathcal{L}\{x''\} + b\mathcal{L}\{x'\} + c\mathcal{L}\{x\} = \mathcal{L}\{f(t)\}$$

Q: What is the L.T. of the derivative of a fcn?
$$\mathcal{L}\{x'\}$$

Integrate by Parts
$du = x'(t)\,dt \quad v = e^{-st}$
$u = x(t) \quad dv = -se^{-st}\,dt$

$$\mathcal{L}\{x'(t)\} = \int_0^\infty e^{-st} x'(t)\,dt =$$

$$= \lim_{b\to\infty}\left[\left(x(t)e^{-st}\right)_0^b - \int_0^b \underbrace{-s}_{\text{constant relative to } dt} e^{-st} x(t)\,dt\right]$$

$$= \lim_{b\to\infty}\left[\left(x(t)e^{-st}\right)_0^b + s\underbrace{\int_0^b e^{-st}x(t)\,dt}_{\mathcal{L}\{x(t)\}} \right]$$

$$= \lim_{b\to\infty}\left[\underbrace{x(b)e^{-sb}}_{\to 0 \text{ if } s>0} - x(0)\overset{1}{\cancel{e^{-s0}}} \right] + s\mathcal{L}\{x(t)\}$$

$$\mathcal{L}\{x'(t)\} = s\mathcal{L}\{x(t)\} - x(0) = \boxed{sX(s) - x(0) \quad \text{if } s>0}$$

Here $\mathcal{L}\{x(t)\} = X(s)$

Notation:

t — lower case

s — upper case for the L.T.

$$x(t) \longrightarrow \mathcal{L} \longrightarrow X(s)$$

$$f(t) \longleftarrow \mathcal{L}^{-1} \longleftarrow F(s)$$

$$\mathcal{L}\{x'(t)\} = sX(s) - x(0)$$

Repeat this for higher order derivatives

$$\mathcal{L}\{x''(t)\} = s\mathcal{L}\{x'(t)\} - x'(0)$$

$$= s\left(sX(s) - x(0)\right) - x'(0)$$

$$= s^2X(s) - sx(0) - x'(0)$$

and so on...

$$\mathcal{L}\{x'''(t)\} = s^3 X(s) - s^2 x(0) - s x'(0) - x''(0)$$

Ex: Apply the L.T. to solve IVP

$$x'' - x' - 6x = 0 \qquad x(0) = 2, \quad x'(0) = -1$$

1. Take the L.T. of both sides

$$\mathcal{L}\{x''\} - \mathcal{L}\{x'\} - 6\mathcal{L}\{x\} = 0$$

$$\left[s^2 X(s) - s x(0) - x'(0)\right] - \left[s X(s) - x(0)\right] - 6X(s) = 0$$

$$\left[s^2 X - 2s + 1\right] - \left[sX - 2\right] - 6X = 0$$

2. Solve for $X(s)$

$$\left[s^2 - s - 6\right] X(s) + \left[-2s + 1 + 2\right] = 0$$

$$(s^2 - s - 6) X(s) = 2s - 3$$

$$X(s) = \frac{2s-3}{s^2 - s - 6} = \frac{2s-3}{(s-3)(s+2)}$$

GOAL: $x(t) = \mathcal{L}^{-1}\{X(s)\} = \mathcal{L}^{-1}\left\{\frac{2s-3}{(s-3)(s+2)}\right\}$

First - rewrite this in more convenient form

3. Expand using method of partial fractions

$$X(s) = \frac{2s-3}{(s-3)(s+2)} = \frac{A}{s-3} + \frac{B}{s+2}$$

A, B are unknown $\longrightarrow$ Find A and B

... both sides by $(s-3)(s+2)$

A, B are unknown

multiply both sides by $(s-3)(s+2)$

$$2s-3 = A(s+2) + B(s-3)$$

collect like terms

$$2s-3 = (A+B)s + (2A-3B)$$

$$0 = (A+B-2)s + (2A-3B+3)$$

$$A+B-2=0$$

$$A = 2-B$$

$$2A-3B+3=0$$

$$2(2-B)-3B+3=0$$

$$4-2B-3B+3=0$$

$$-5B=-7$$

$$B=\frac{7}{5}$$

$$A = 2-\frac{7}{5} = \frac{10-7}{5} = \frac{3}{5}$$

$$X(s) = \frac{A}{s-3} + \frac{B}{s+2} = \left(\frac{3}{5}\right)\left(\frac{1}{s-3}\right) + \left(\frac{7}{5}\right)\left(\frac{1}{s+2}\right)$$

4. Take the inverse L.T.

$$x(t) = \mathcal{L}^{-1}\{X(s)\} = \frac{3}{5}\mathcal{L}^{-1}\left\{\frac{1}{s-3}\right\} + \frac{7}{5}\mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\}$$

look up in table

$$\boxed{x(t) = \frac{3}{5}e^{3t} + \frac{7}{5}e^{-2t}}$$

☆ Laplace Transform Procedure:

variable t

variable s

variable t | variable s

ODE for $x(t)$ $\longrightarrow$ $\mathcal{L}$ $\longrightarrow$ algebraic eqn for $X(s)$

$\downarrow$

solve for $X(s)$ using algebra, Partial Fractions

$\downarrow$

$x(t)$ $\longleftarrow$ $\mathcal{L}^{-1}$ $\longleftarrow$ $X(s)$

NOTE: The L.T. moves us back and forth between t variable and the s variable

The L.T. transforms the ODE in t into and algebraic eqn in s

Q: What does s represent?

A: Think of s as a "frequency"

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$$

if $s = a + ib$

$f(t) \longrightarrow F(s)e^{-(a+ib)t}$

$F(s)e^{-at}$ dissipation

$F(s)e^{-ibt}$ – frequency

say that s is a "frequency"

inspired by Fourier Transform

$$\text{F.T.} = \int_0^\infty e^{ikt} f(t)\,dt \longrightarrow$$

t – time

k – frequency

F.T. $= \int_0^{\infty} e^{ikt} f(t)\,dt$

II. Transform Theorems:

Before we showed that

t	s
derivative in t	multiplication by s

$$\frac{d}{dt}\left(e^{st}\right) = s\left(e^{st}\right)$$

Thm 1: (Transform of Derivatives)

$$\mathcal{L}\{x'(t)\} = sX(s) - x(0)$$

Similarly

Thm 2: (Transforms of Integrals)

$$\mathcal{L}\left\{\int_0^t f(\tau)\,d\tau\right\} = \frac{1}{s}\mathcal{L}\{f(t)\} = \frac{F(s)}{s}$$

and conversely

$$\mathcal{L}^{-1}\left\{\frac{F(s)}{s}\right\} = \int_0^t f(\tau)\,d\tau$$

t	s
integral in t	division by s

$$\int e^{st}\,dt = \frac{e^{st}}{s}$$

We can use Thm 2 to help us find inverse L.T.

Ex: Find the inverse L.T. of $G(s) = \dfrac{1}{s(s-3)}$

Thm 2: $\mathcal{L}^{-1}\left\{\dfrac{F(s)}{s}\right\} = \int_0^t f(\tau)\,d\tau$

Need to find $F(s)$ and $f(t)$

$$G(s) = \frac{1}{s(s-3)} = \frac{F(s)}{s} \longrightarrow F(s) = \frac{1}{s-3}$$

$$f(t) = \mathcal{L}^{-1}\{F(s)\} = e^{3t}$$

$$g(t) = \mathcal{L}^{-1}\{G(s)\} = \mathcal{L}^{-1}\left\{\frac{F(s)}{s}\right\} \overset{\text{Thm 2}}{=} \int_0^t f(\tau)\,d\tau$$

$$\int^t e^{3\tau} \ldots = \left[e^{3\tau} \right]^t$$

$$= \int_0^t e^{3\tau}\, d\tau = \left[\frac{e^{3\tau}}{3} \right]_0^t$$

$$\boxed{g(t) = \frac{1}{3}\left[e^{3t} - 1 \right]}$$

another way:

$$g(t) = \mathcal{L}^{-1}\left\{ \frac{1}{s(s-3)} \right\} = \int_0^t \mathcal{L}^{-1}\left\{ \frac{1}{s-3} \right\} d\tau$$

$$= \int_0^t e^{3\tau}\, d\tau = \cdots = \frac{1}{3}\left[e^{3t} - 1 \right]$$

III. Systems of ODE:

Ex: $\begin{cases} x' = 2x + y \\ y' = 6x + 3y \end{cases}$ $\quad x(0) = 1$, $\quad y(0) = -2$

1. Take the L.T. of both sides

$$\mathcal{L}\{x'\} = 2\mathcal{L}\{x\} + \mathcal{L}\{y\}$$

$$\mathcal{L}\{y'\} = 6\mathcal{L}\{x\} + 3\mathcal{L}\{y\}$$

$$sX - 1 = sX(s) - x(0) = 2X + Y$$

$$sY + 2 = sY(s) - y(0) = 6X + 3Y$$

2. Solve for $X(s)$ and $Y(s)$

collect like terms ← solve this eqn for Y

$$\begin{cases} (s-2)X - Y = 1 \\ \ldots X + (s-3)Y = -2 \end{cases}$$

$$\begin{cases} (s-2)X - Y = 1 \\ -6X + (s-3)Y = -2 \end{cases}$$

plug into 2nd eqn

$$(s-2)X - 1 = Y$$

$$-6X + (s-3)\left[(s-2)X - 1\right] = -2$$

$$\left[(s-3)(s-2) - 6\right]X = -2 + s - 3 = s - 5$$

$$\left[s^2 - 5s + \cancel{6} - \cancel{6}\right]X = s - 5$$

$$s(s-5)X = s - 5$$

$$\boxed{X(s) = \frac{1}{s}}$$

$$Y(s) = (s-2)X - 1 = (s-2)\left(\frac{1}{s}\right) - (1)\left(\frac{s}{s}\right) = \frac{s-2-s}{s}$$

$$\boxed{Y(s) = \frac{-2}{s}}$$

3. Take the inverse L.T.

$$x(t) = \mathcal{L}^{-1}\left\{\frac{1}{s}\right\} = 1$$

$$y(t) = \mathcal{L}^{-1}\left\{\frac{-2}{s}\right\} = -2$$

solutions: $\boxed{x(t) = 1 \quad \text{and} \quad y(t) = -2}$

Ex: Apply Thm 2 to find the inverse L.T. of

$$F(s) = \frac{1}{s^2(s^2-1)}$$

Thm 2: $\mathcal{L}^{-1}\left\{\frac{F(s)}{s}\right\} = \int_0^t f(\tau)\,d\tau$

let's write this as:

$$\mathcal{L}^{-1}\left\{\frac{F(s)}{s}\right\} = \int_0^t \mathcal{L}^{-1}\{F(s)\}\,d\tau$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2(s^2-1)}\right\} \overset{\text{Thm 2}}{=} \int_0^t \mathcal{L}^{-1}\left\{\frac{1}{s(s^2-1)}\right\} d\tau$$

$$\overset{\text{Thm 2 again}}{=} \int_0^t \int_0^\tau \mathcal{L}^{-1}\left\{\frac{1}{s^2-1}\right\} dt\, d\tau$$

$\mathcal{L}^{-1}\left\{\frac{1}{s^2-1}\right\} = \sinh(t)$

look up in table of L.T.

$$= \int_0^t \left(\int_0^\tau \sinh(t)\,dt\right) d\tau$$

$$= \int_0^t \Big[\cosh(t)\Big]_0^\tau d\tau$$

$$= \int_0^t \left[\cosh(\tau) - \cancelto{1}{\cos(0)}\right] d\tau$$

$$= \Big[\sinh(\tau) - \tau\Big]_0^t$$

$$= \sinh(t) - t - \cancelto{0}{\sinh(0)} + \cancelto{0}{0}$$

$$\boxed{\mathcal{L}^{-1}\left\{\frac{1}{s^2(s^2-1)}\right\} = \sinh(t) - t}$$

NOTE could also solve this using partial fractions

☆ Summary:

- Transforms of Derivatives

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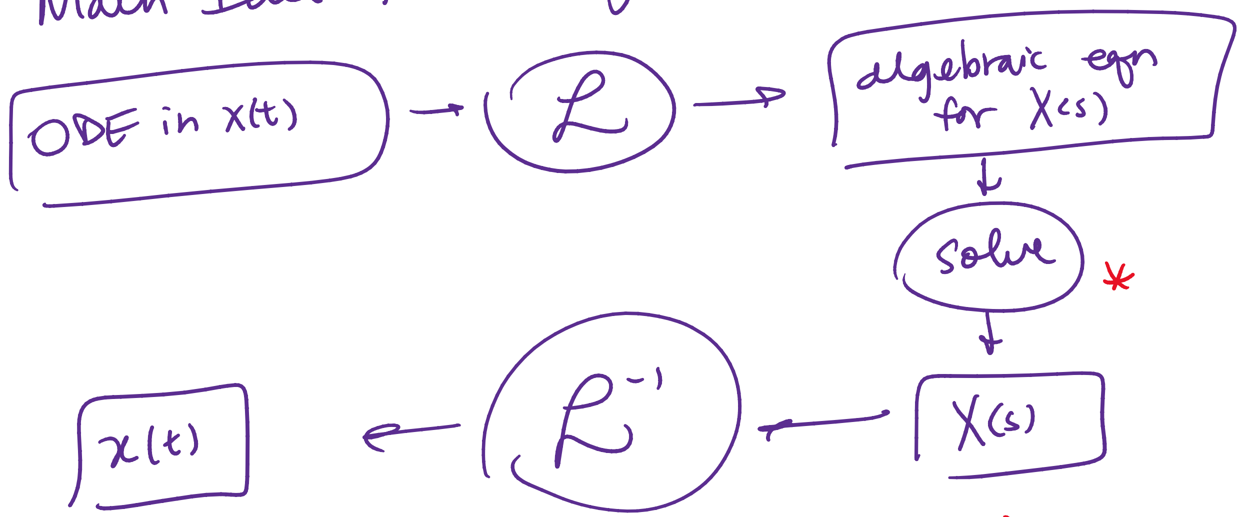
$$\mathcal{L}\{x'(t)\} = sX(s) - x(0)$$

- Transforms of Integrals

$$\mathcal{L}\left\{\int_0^t f(\tau)\,d\tau\right\} = \frac{F(s)}{s}$$

$$\mathcal{L}^{-1}\left\{\frac{F(s)}{s}\right\} = \int_0^t f(\tau)\,d\tau$$

- Main Idea for solving ODEs:



- Often use Partial fractions to solve* for X(s)