Lecture 28

Section 7.3:

Translation & Partial Fractions

Announcements:

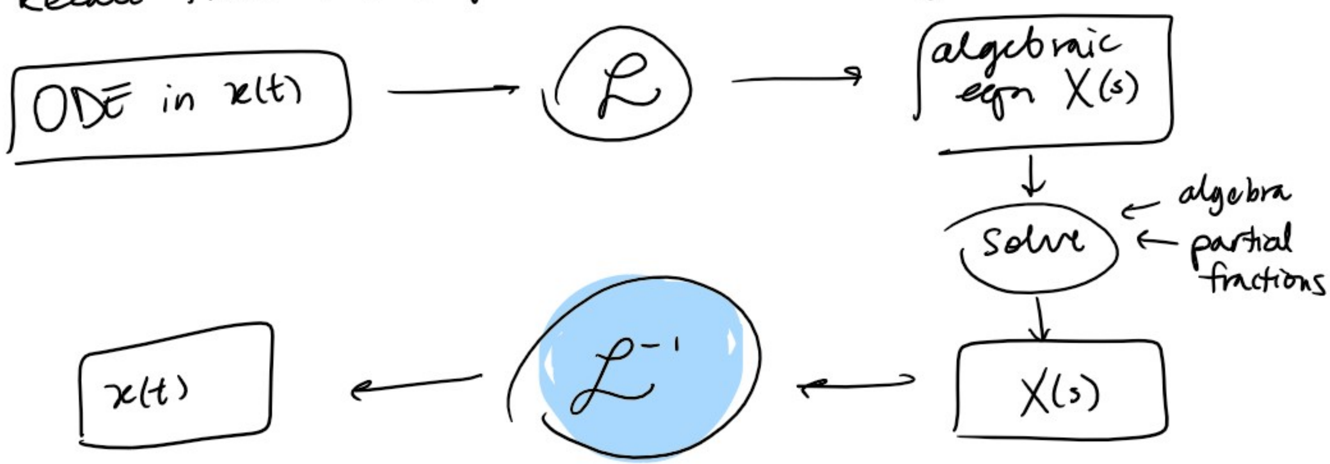
- Office Hours Today @ 2:30–3:30pm
- Final Exam Friday Aug 6 @ 8am–10am
  - Evening Thurs Aug 5 @ 6pm–8pm
- Review in class Tues Aug 3

Warm up:

Fill in the Table of Laplace Transforms below:

$f(t)$	$F(s)$	
e^{at}	$\frac{1}{s-a}$	$s>a$
t^n ($n \geq 0$ integer)	$\frac{n!}{s^{n+1}}$	$s>0$
$\cos(kt)$	$\frac{s}{s^2+k^2}$	$s>0$

Recall from last, procedure for solving ODEs



KEY STEP: arrange $X(s)$ so that we can take $\mathcal{L}^{-1}\{X(s)\} = x(t)$

I. Translation:

This property helps us take $\mathcal{L}^{-1}\{X(s)\}$

$F(s) = \mathcal{L}\{f(t)\}$

Thm (Translation on the s-axis)

$$\mathcal{L}\{e^{at} f(t)\} = F(s-a)$$

t	s
multiply by e^{at}	then $s \mapsto s-a$

and conversely

$\mathcal{L}\{ t \ldots \}$

and conversely

$$\mathcal{L}^{-1}\{F(s-a)\} = e^{at} f(t)$$

multiply by e^{at} | then $s \longmapsto s-a$

$$e^{at} e^{-st} = e^{-(s-a)t}$$

<u>Ex</u>: Find the L.T. of $g(t) = e^{3t}\cos(\pi t)$

$a = 3$

$$\mathcal{L}\{e^{3t}\cos(\pi t)\} \overset{\checkmark \text{ Thm}}{=} F(s-3)$$

$f(t) = \cos(\pi t)$ $\qquad F(s) = \mathcal{L}\{\cos(\pi t)\} = \dfrac{s}{s^2+\pi^2}$

$$\boxed{\mathcal{L}\{g(t)\} = \frac{(s-3)}{(s-3)^2+\pi^2}}$$

Table of Laplace Transforms

$f(t)$	$F(s)$	
$e^{at} t^n$	$\dfrac{n!}{(s-a)^{n+1}}$	$s > a$
$e^{at}\cos(kt)$	$\dfrac{s-a}{(s-a)^2+k^2}$	$s > a$
$e^{at}\sin(kt)$	$\dfrac{k}{(s-a)^2+k^2}$	$s > a$

<u>Ex</u>: Find the inverse L.T. of $G(s) = \dfrac{2s+3}{s^2+2s+5}$

Notice that: $s^2+2s+5 = (s+1)^2 + 2^2$

$= (s^2+2s+1) + 4$

$$G(s) = \frac{2s}{(s+1)^2+2^2} + \frac{3}{(s+1)^2+2^2}$$

$$= \frac{2(s+1)}{(s+1)^2+2^2} + \frac{1}{(s+1)^2+2^2}$$

from table

$$= 2\left[\frac{s+1}{(s+1)^2+2^2}\right] + \frac{1}{2}\left[\frac{2}{(s+1)^2+2^2}\right]$$

$$\boxed{g(t) = \mathcal{L}^{-1}\{G(s)\} = 2e^{-t}\cos(2t) + \frac{1}{2}e^{-t}\sin(2t)}$$

II. Partial Fractions:

Often our solution $X(s)$ is in the form

$X(s) = \frac{P(s)}{Q(s)}$ called a rational function

where $P(s)$ and $Q(s)$ are both polynomials and $\deg(P(s)) <$ degree of $Q(s)$

GOAL: Expand $X(s) = \frac{P(s)}{Q(s)}$ using Partial Fractions

Rule 1: (Linear Factors)

$$\frac{P(s)}{(s-a)^n} = \frac{A_1}{s-a} + \frac{A_2}{(s-a)^2} + \dots + \frac{A_n}{(s-a)^n}$$

$\mathcal{L}^{-1}\{\cdot\}$

$\frac{A_1}{s-a} \mapsto e^{at}$

$\frac{A_n}{(s-a)^n} \mapsto C_n e^{at} t^{n-1}$

Rule 2: (Quadratic Factors)

$$\frac{P(s)}{[(s-a)^2+b^2]^n} = \frac{A_1 s + B_1}{(s-a)^2+b^2} + \frac{A_2 s + B_2}{[(s-a)^2+b^2]^2} + \dots + \frac{A_n s + B_n}{[(s-a)^2+b^2]^n}$$

$\mathcal{L}^{-1}\{\cdot\}$ $C_1 e^{at}\cos(bt) + C_2 e^{at}\sin(bt)$ derive this next class

Ex: Find the inverse L.T. of $G(s) = \frac{5}{s^4+9s^2}$

$$G(s) = \frac{5}{s^4+9s^2} = \frac{5}{s^2(s^2+9)}$$

$$G(s) = \frac{5}{s^4 + 9s^2} = \frac{5}{\underbrace{s^2}_{\text{linear}}\underbrace{(s^2+9)}_{\text{quadratic}}}$$

Rule 1: $\frac{1}{s^2} \longmapsto \frac{A}{s} + \frac{B}{s^2}$

Rule 2: $\frac{1}{s^2+9} \longmapsto \frac{Cs+D}{s^2+9}$

Partial Fractions: Find A, B, C, D

$$\frac{5}{s^2(s^2+9)} = \frac{A}{s} + \frac{B}{s^2} + \frac{Cs+D}{s^2+9}$$

multiply by common denom

$$5 = As(s^2+9) + B(s^2+9) + (Cs+D)s^2$$

Evaluate s @ key points

$\boxed{@\ s=0}$ $s=0,\quad s^2=0 \quad s^2+9=9$

$$5 = \cancel{A\cdot 0(9)}^{0} + B(9) + \cancel{(C\cdot 0 + D)0}^{0} \rightarrow B = \frac{5}{9}$$

$\boxed{@\ s=3i}$ $s^2+9=0 \quad s=3i \quad s^2=-9$

$$5 = \cancel{A(3i)(0)}^{0} + \cancel{B\cdot 0}^{0} + (C(3i)+D)(-9)$$

$$0i + 5 = -27iC - 9D$$

$$5 = -9D \rightarrow D = -\frac{5}{9}$$

$$0 = 27C$$

$C = 0$

$\boxed{@\ s=1}$ $s^2=1 \quad s^2+9=10$

$$5 = A(1)(10) + B(10) + (\cancel{C(1)}^{0}+D)(1)$$

$$\cancel{5} = 10A + \frac{5}{9}(10) - \frac{5}{9} = 10A + \frac{45}{9} = 10A + \cancel{5}$$

$\rightarrow A = 0$

$$G(s) = \frac{5}{s^2(s^2+9)} = \cancel{\frac{A}{s}}^{0} + \frac{B}{s^2} + \frac{\cancel{Cs}^{0}+D}{s^2+9}$$

$$G(s) = \frac{5}{s^2(s^2+9)} = \frac{\cancel{A}}{s} + \frac{B}{s^2} + \frac{\cancel{C}s+D}{s^2+9}$$

$$= \frac{5}{9}\left(\frac{1}{s^2}\right) + \left(\frac{-5}{9}\right)\left(\frac{1}{s^2+9}\right)$$

$$= \frac{5}{9}\left(\frac{1}{s^2}\right) - \left(\frac{5}{27}\right)\left(\frac{3}{s^2+9}\right)$$

$$g(t) = \mathcal{L}^{-1}\{G(s)\} = \frac{5}{9}\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} - \frac{5}{27}\mathcal{L}^{-1}\left\{\frac{3}{s^2+9}\right\}$$

$$\boxed{g(t) = \frac{5}{9}t - \frac{5}{27}\sin(3t)}$$

Ex: Use L.T. to solve the IVP:

$$\begin{cases} y^{(4)} + 2y'' + y = 4te^t \\ y(0) = y'(0) = y''(0) = y'''(0) = 0 \end{cases}$$

1. Take the L.T. of both sides:

$$\mathcal{L}\{y^{(4)}(x)\} = s^4 Y(s) - s^3 \cancel{y(0)}^{\,0} - s^2 \cancel{y'(0)}^{\,0} - s\,\cancel{y''(0)}^{\,0} - \cancel{y'''(0)}^{\,0}$$

$$\mathcal{L}\{y''(x)\} = s^2 Y(s) - s\cancel{y(0)}^{\,0} - \cancel{y'(0)}^{\,0}$$

$$\mathcal{L}\{4te^t\} = 4\mathcal{L}\{te^t\} = 4\left(\frac{1}{(s-1)^2}\right) = \frac{4}{(s-1)^2}$$

$$s^4 Y(s) + 2s^2 Y(s) + Y(s) = \frac{4}{(s-1)^2}$$

$$(s^4 + 2s^2 + 1)Y(s) = \frac{4}{(s-1)^2}$$

$$(s^2+1)^2 Y(s) = \frac{4}{(s-1)^2}$$

$$Y(s) = \frac{4}{(s-1)^2[s^2+1]^2}$$

$$Y(s) = \frac{4}{(s-1)^2[s^2+1]^2}$$

2. Expand using Partial Fractions:

$$\frac{4}{(s-1)^2[s^2+1]^2} = \frac{A}{(s-1)^2} + \frac{B}{(s-1)} + \frac{Cs+D}{[s^2+1]^2} + \frac{Es+F}{[s^2+1]}$$

multiply both sides by common denom.
evaluate at key points (@ $s=1$, $s=i$)

From that: $A=1$ $D=0$
$B=-2$ $E=2$
$C=2$ $F=1$

$$Y(s) = \frac{1}{(s-1)^2} + \frac{-2}{s-1} + \frac{2s}{(s^2+1)^2} + \frac{2s+1}{s^2+1}$$

$$= \underbrace{\frac{1}{(s-1)^2}}_{\mathcal{L}\{e^t t\}} - 2\underbrace{\left[\frac{1}{s-1}\right]}_{\mathcal{L}\{e^t\}} + 2\underbrace{\left[\frac{s}{(s^2+1)^2}\right]}_{\substack{\text{derive this} \\ \text{next class} \\ \mathcal{L}\{\frac{t}{2}\sin(t)\}}} + 2\underbrace{\left[\frac{s}{s^2+1}\right]}_{\mathcal{L}\{\cos(t)\}} + \underbrace{\left[\frac{1}{s^2+1}\right]}_{\mathcal{L}\{\sin(t)\}}$$

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = te^t - 2e^t + t\sin(t) + 2\cos(t) + \sin(t)$$

$$\boxed{y(t) = (t-2)e^t + (t+1)\sin(t) + 2\cos(t)}$$

Ex: Find the inverse L.T. of $G(s) = \dfrac{60}{(s^2+4)((s+3)^2+25)}$

Partial Fractions:

$$\frac{60}{(s^2+4)((s+3)^2+25)} = \frac{As+B}{s^2+4} + \frac{Cs+D}{(s+3)^2+25}$$

$$\frac{60}{(s^2+4)((s+3)^2+25)} = \frac{As+B}{s^2+4} + \frac{Cs+D}{(s+3)^2+25}$$

multiply by common denom.

$$60 = (As+B)[(s+3)^2+25) + (Cs+D)[s^2+4]$$

Evaluate at key points:

$\boxed{@s = -3+5i}$ $\quad (s+3)^2+25 = (\cancel{-3}+5i\cancel{+3})^2+25 = -25+25 = 0$

$$60 = \cancel{(As+B)}(0) + [C(-3+5i)+D][(-3+5i)^2+4]$$

$$= [-3C+5iC+D][9-15i-15i-25+4]$$

$$= [-3C+5iC+D][-12-30i]$$

$$= (-3+5i)(-12-30i)C + (-12-30i)D$$

$$= (36 + 90i - 60i + 150)C + (-12-30i)D$$

$$60 = (186 - \overset{+30}{\cancel{150}}i)C + (-12-30i)D$$

$$= (186C - 12D) + i(\overset{+30}{\cancel{-150}}C - 30D)$$

$$60 = 186C - 12D$$

$$0 = -\overset{+30}{\cancel{150}}C - 30D$$

$$\cancel{5C = \frac{150C}{30} = D} \qquad C = D$$

$$60 = 174D$$

$$\boxed{C = D = \frac{60}{174} = \frac{10}{29}}$$

similarly find

$$A = \frac{-10}{29} \qquad B = \frac{50}{29}$$

solution

$$g(t) = \frac{5}{29}\left[-2\cos(2t) + 5\sin(2t)\right] + \frac{2}{29}e^{-3t}\left[5\cos(5t) - 2\sin(5t)\right]$$

☆ Summary:

- Translation on the s-axis

$$\mathcal{L}\{e^{at}f(t)\} = F(s-a)$$

$$\mathcal{L}^{-1}\{F(s-a)\} = e^{at}f(t)$$

- Partial Fractions

Rule 1: $$\frac{P(s)}{(s-a)^n} = \frac{A_1}{s-a} + \frac{A_2}{(s-a)^2} + \dots + \frac{A_n}{(s-a)^n}$$

Rule 2: $$\frac{P(s)}{[(s-a)^2+b^2]^n} = \frac{A_1s+B_1}{(s-a)^2+b^2} + \dots + \frac{A_ns+B_n}{[(s-a)^2+b^2]^n}$$