Lecture 29

Section 7.4: Derivatives, Integrals, and Products of Transforms

Announcements:
- Online HW + A7 due Tues Aug 3
- Final Exam Fri Aug 6 @ 8am–10am
- Evening Exam Thurs Aug 5 @ 6pm–8pm

Warm up: Which of the following statements are true about the Laplace Transform? (check all that apply)

(a) $\mathcal{L}\{af(t)\} = aF(s)$ ✓

(b) $\mathcal{L}\{[f(t)]^2\} = [F(s)]^2$ X

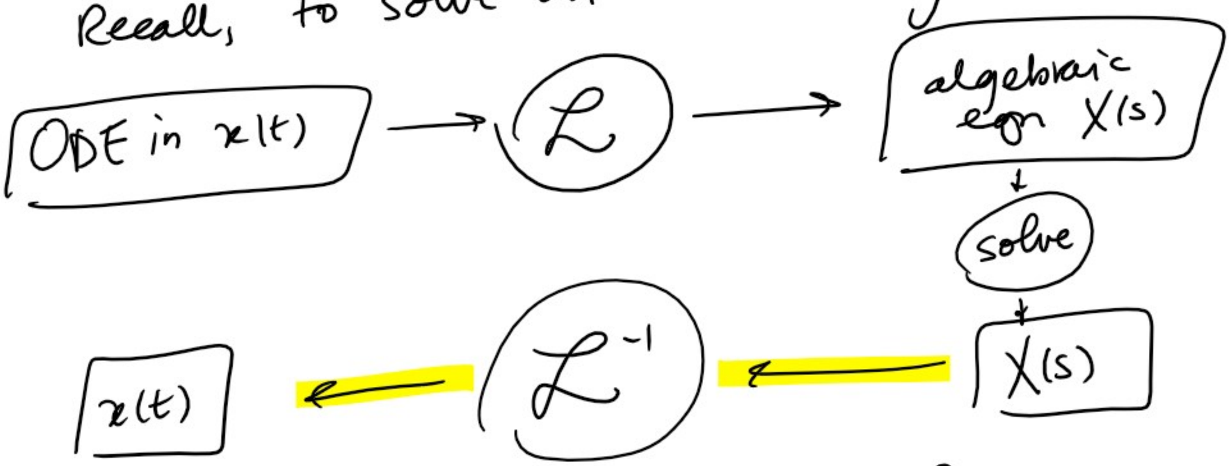
(c) $\mathcal{L}\{f(t)+g(t)\} = F(s)+G(s)$ ✓

(d) $\mathcal{L}\{f(t)g(t)\} = F(s)G(s)$ X

linearity

I. Convolutions:

Recall, to solve an ODE using L.T.



our soln $X(s)$ often has the form

$$X(s) = F(s)\,G(s)$$

want to find

$$x(t) = \mathcal{L}^{-1}\{F(s)\,G(s)\}$$

It's NOT true that

$$\mathcal{L}^{-1}\{F(s)\,G(s)\} \cancel{=} f(t)\,g(t)$$

$$\mathcal{L}^{-1}\{F(s)\,G(s)\} \cancel{=} f(t)\,g(t)$$

Def: The convolution of two functions $f(t)$ and $g(t)$ is defined

$$\left.\begin{aligned}(f*g)(t) &= \int_0^t f(\tau)\,g(t-\tau)\,d\tau \\ &= \int_0^t f(t-\tau)\,g(\tau)\,d\tau\end{aligned}\right\}\ \text{order of fcns doesn't matter}$$

Thm: (Convolution Property)

$$\mathcal{L}\{(f*g)(t)\} = \mathcal{L}\{f(t)\}\,\mathcal{L}\{g(t)\} = F(s)G(s)$$

and conversely

$$\mathcal{L}^{-1}\{F(s)G(s)\} = (f*g)(t)$$

Ex: Find the inverse L.T. of $H(s) = \dfrac{2}{(s+1)(s-3)}$

(NOTE: Could use partial fractions

$$\frac{2}{(s+1)(s-3)} = \frac{A}{s+1} + \frac{B}{s-3}$$

Find A and B)

Instead, use convolution property

$$H(s) = \underbrace{\left(\frac{1}{s+1}\right)}_{F(s)}\underbrace{\left(\frac{2}{s-3}\right)}_{G(s)}$$

$$f(t) = \mathcal{L}^{-1}\{F(s)\} = e^{-t}$$

$$g(t) = \mathcal{L}^{-1}\{G(s)\} = 2e^{3t}$$

$$\mathcal{L}^{-1}\{H(s)\} = \mathcal{L}^{-1}\{F(s)G(s)\} \overset{\text{Thm}}{=} (f*g)(t)$$

$$= \int_0^t f(\tau)\,g(t-\tau)\,d\tau$$

$$= \int_0^t e^{-\tau}\left(2e^{3(t-\tau)}\right)d\tau = \int_0^t 2e^{3t-4\tau}\,d\tau$$

$$= \int_0^t e^{-\tau}\left(2e^{3(t-\tau)}\right) d\tau = \int_0^t 2e^{3t-4\tau} \, d\tau$$

$$= 2e^{3t} \int_0^t e^{-4\tau} d\tau = 2e^{3t} \left[\frac{e^{-4\tau}}{-4} \right]_0^t$$

$$= 2e^{3t} \left[\frac{e^{-4t} - 1}{-4} \right] = -\frac{1}{2}\left[e^{-t} - e^{3t} \right]$$

$$\boxed{\mathcal{L}^{-1}\{H(s)\} = \frac{1}{2}\left[e^{3t} - e^{-t} \right]}$$

use partial fractions to check

NOTE! We can also use the convolution property to write down solutions to ODEs.

Ex: Apply the convolution property to write down a solution to:

$$x'' + 4x = f(t) \qquad x(0) = x'(0) = 0$$

1. Take the L.T. of both sides

$$\left[s^2 X(s) - s \cdot 0 - 0\right] + 4\left[X(s)\right] = F(s)$$

$$(s^2+4)\, X(s) = F(s)$$

$$X(s) = \frac{F(s)}{s^2+4} = F(s)\left(\frac{1}{s^2+4}\right)$$

2. Take the inverse L.T.

$$G(s) = \frac{1}{s^2+4} \qquad g(t) = \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{2}{s^2+4}\right\} = \frac{1}{2}\sin(2t)$$

$$x(t) = \mathcal{L}^{-1}\{F(s)\,G(s)\} = (f * g)(t)$$

$$x(t) = \mathcal{L}^{-1}\{F(s)G(s)\}$$

$$= \int_0^t f(t-\tau)g(\tau)\,d\tau$$

$$\boxed{x(t) = \frac{1}{2}\int_0^t f(t-\tau)\sin(2\tau)\,d\tau}$$

works for any nonzero $f(t)$

II. Differentiation:

Recall: Transforms of Derivatives

$$\mathcal{L}\{x'(t)\} = sX(s) - x(0)$$

t	s
deriv in t	mult by s

The reverse is "almost true"

Thm (Differentiation)

$$\mathcal{L}\{-t\,f(t)\} = F'(s) = \frac{dF}{ds}$$

and conversely

$$f(t) = \mathcal{L}^{-1}\{F(s)\} = \frac{-1}{t}\mathcal{L}^{-1}\{F'(s)\}$$

t	s
mult by $-t$	deriv in s

Ex: Find the L.T. of $g(t) = t\cosh(2t)$

here, let $f(t) = -\cosh(2t)$

$$\mathcal{L}\{g(t)\} = \mathcal{L}\{-t\,f(t)\} \overset{\text{Thm}}{=} \frac{dF}{ds}$$

$$= \frac{d}{ds}\mathcal{L}\{-\cosh(2t)\} = \frac{d}{ds}\left[\frac{-s}{s^2-4}\right]$$

$$= (-1)\left(\frac{1}{s^2-4}\right) + (\cancel{+}s)(\cancel{+}1)(s^2-4)^{-2}(2s)$$

$$= \frac{-1}{s^2-4}\frac{(s^2-4)}{(s^2-4)} + \frac{2s^2}{(s^2-4)^2} = \frac{-s^2+4+2s^2}{(s^2-4)^2}$$

$$\boxed{\mathcal{L}\{g(t)\} = \frac{s^2+4}{[s^2-4]^2}}$$

III. Integration !

Recall: Transform of integrals

$$\mathcal{L}\left\{\int_0^t f(\tau)\,d\tau\right\} = \frac{F(s)}{s}$$

t	s
integrate in t	dividing by s

The reverse is "almost true"

Thm : (Integration of Transforms)

$$\mathcal{L}\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(\sigma)\,d\sigma$$

note indefinite integral

t	s
divide by t	integrate in s

and conversely

$$f(t) = \mathcal{L}^{-1}\{F(s)\} = t\,\mathcal{L}^{-1}\left\{\int_s^\infty F(\sigma)\,d\sigma\right\}$$

Ex: Find $\mathcal{L}\left\{\frac{\sinh(t)}{t}\right\}$

$$\mathcal{L}\left\{\frac{\sinh(t)}{t}\right\} = \mathcal{L}\left\{\frac{f(t)}{t}\right\} \overset{\text{Thm}}{=} \int_s^\infty F(\sigma)\,d\sigma$$

$f(t) = \sinh(t)$

$F(s) = \mathcal{L}\{\sinh(t)\} = \frac{1}{s^2-1}$

$$= \int_s^\infty \frac{1}{\sigma^2-1}\,d\sigma = \int_s^\infty \frac{d\sigma}{(\sigma-1)(\sigma+1)}$$

partial fractions

$$\frac{1}{(\sigma-1)(\sigma+1)} = \frac{A}{\sigma-1} + \frac{B}{\sigma+1} = \frac{1}{2}\left(\frac{1}{\sigma-1}\right) - \frac{1}{2}\left(\frac{1}{\sigma+1}\right)$$

$\ldots\ldots \quad A = \frac{1}{2} \quad B = -\frac{1}{2}$

$$\int^\infty \frac{d\sigma}{} \quad \frac{1}{}\int^\infty \frac{d\sigma}{}$$

$$= \frac{1}{2}\int_s^\infty \frac{d\sigma}{\sigma-1} - \frac{1}{2}\int_s^\infty \frac{d\sigma}{\sigma+1}$$

$$= \frac{1}{2}\lim_{b\to\infty}\left[\int_s^b \frac{d\sigma}{\sigma-1} - \int_s^b \frac{d\sigma}{\sigma+1}\right]$$

$$= \frac{1}{2}\lim_{b\to\infty}\left[\left(\ln(\sigma-1) - \ln(\sigma+1)\right)\Big|_s^b\right]$$

$$= \frac{1}{2}\lim_{b\to\infty}\left[\ln(b-1) - \ln(s-1) - \ln(b+1) + \ln(s+1)\right]$$

$$= \frac{1}{2}\lim_{b\to\infty}\left[\underbrace{\cancel{\ln\left(\underbrace{\frac{b-1}{b+1}}_{\xrightarrow[b\to\infty]{} 1}\right)}}_{\xrightarrow[b\to\infty]{} 0} + \ln\left(\frac{s+1}{s-1}\right)\right]$$

$$\boxed{\mathcal{L}\left\{\frac{\sinh(t)}{t}\right\} = \frac{1}{2}\ln\left(\frac{s+1}{s-1}\right)}$$

NOTE: Need to use the Integral Property to take the inverse L.T. from partial fractions Rule 2

Rule 2: $$\frac{P(s)}{[s^2+b^2]^n} = \frac{A_1 s + B_1}{s^2+b^2} + \frac{A_2 s + B_2}{[s^2+b^2]^2} + \dots + \frac{A_n s + B_n}{[s^2+b^2]^n}$$

$$\frac{A_2 s + B_2}{[s^2+b^2]^2} = \underbrace{\frac{A_2 s}{[s^2+b^2]^2}}_{\text{first}} + \frac{B_2}{[s^2+b^2]^2}$$

Ex: Find $\mathcal{L}^{-1}\left\{\frac{2s}{[s^2+1]^2}\right\} \overset{\text{Thm}}{=} t\,\mathcal{L}^{-1}\left\{\int_s^\infty \frac{2\sigma\, d\sigma}{[\sigma^2+1]^2}\right\}$

Ex: Find $\mathcal{L}^{-1}\left\{ \frac{2s}{[s^2+1]^2} \right\} = t\mathcal{L}^{-1}\left\{ \int_s^{\infty} \frac{2\sigma\, d\sigma}{[\sigma^2+1]^2} \right\}$

$u = \sigma^2+1$
$du = 2\sigma\, d\sigma$

$$= t\mathcal{L}^{-1}\left\{ \int_{s^2+1}^{\infty} \frac{du}{u^2} \right\}$$

$$= t\mathcal{L}^{-1}\left\{ \lim_{b\to\infty} \int_{s^2+1}^{b} \frac{du}{u^2} \right\}$$

$$= t\mathcal{L}^{-1}\left\{ \lim_{b\to\infty} \left[-\frac{1}{u} \right]_{s^2+1}^{b} \right\}$$

$$= t\mathcal{L}^{-1}\left\{ \lim_{b\to\infty} \left[\underbrace{-\frac{1}{b}}_{\to 0 \text{ as } b\to\infty} + \frac{+1}{s^2+1} \right] \right\}$$

$$= t\mathcal{L}^{-1}\left\{ \frac{1}{s^2+1} \right\} = \boxed{t\sin(t)}$$

Ex: 2nd term from rule 2 in partial fractions

$$\mathcal{L}^{-1}\left\{ \frac{B_2}{[s^2+b^2]^2} \right\}$$

Ex: Find $\mathcal{L}^{-1}\left\{ \frac{1}{(s^2+4)^2} \right\} = \mathcal{L}^{-1}\left\{ \frac{1}{4} \underbrace{\left(\frac{2}{s^2+4}\right)}_{\mathcal{L}\{\sin(2t)\}} \left(\frac{2}{s^2+4}\right) \right\}$

convolution property $\rightarrow$ $= \frac{1}{4}(f*g)$

$f = \sin(2t)$
$g = \sin(2t)$

$$= \frac{1}{4}\int_0^t \sin(2(t-\tau))\sin(2\tau)\,d\tau$$

$$= \frac{1}{4}\int_0^t \sin(2t-2\tau)\sin(2\tau)\,d\tau$$

$\sin(A)\sin(B) = \cos(A-B) - \cos(A+B)$

$-\frac{1}{4}\int_0$ sin ...

Use trig formula: $2\sin(A)\sin(B) = \cos(A-B) - \cos(A+B)$

Let $A = 2t - 2\tau \qquad B = 2\tau$

$$= \frac{1}{4}\cdot\frac{1}{2}\int_0^t \left[\cos(2t-2\tau-2\tau) - \cos(2t-\cancel{2\tau}+\cancel{2\tau})\right] d\tau$$

$$= \frac{1}{8}\int_0^t \left[\cos(2t-4\tau) - \underbrace{\cos(2t)}_{\text{has no } \tau \text{ dependence}}\right] d\tau$$

$$= \frac{1}{8}\left[\frac{\sin(2t-4\tau)}{-4} - \cos(2t)\,\tau\right]_0^t$$

$$= \frac{1}{8}\left[\frac{\sin(2t-4t) - \sin(2t-0)}{-4} - \cos(2t)(t-0)\right]$$

$\sin(-2t) = -\sin(2t)$

$$= \frac{1}{8}\left[\frac{\sin(-2t) - \sin(2t)}{-4} - t\cos(2t)\right]$$

$$= \frac{1}{8}\left[\frac{+\cancel{2}\sin(2t)}{+\cancel{4}\,2} - t\cos(2t)\right]$$

$$\boxed{\mathcal{L}\left\{\frac{1}{(s^2+4)^2}\right\} = \frac{1}{16}\left[\sin(2t) - 2t\cos(2t)\right]}$$

☆ Summary:

- Convolution Property

$$\mathcal{L}^{-1}\{F(s)G(s)\} = (f*g)(t) = \int_0^t f(t-\tau)\,g(\tau)\,d\tau$$

- Differentiation

$$\mathcal{L}\{-t f(t)\} = \frac{dF}{ds}$$

- Differentiation:

$$\mathcal{L}\{-t f(t)\} = \frac{dF}{ds}$$

$$\mathcal{L}^{-1}\{F(s)\} = -\frac{1}{t}\mathcal{L}^{-1}\left\{\frac{dF}{ds}\right\}$$

- Integration:

$$\mathcal{L}\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(\sigma)\,d\sigma$$

$$\mathcal{L}^{-1}\{F(s)\} = t\,\mathcal{L}^{-1}\left\{\int_s^\infty F(\sigma)\,d\sigma\right\}$$