Lecture 2

Section 5.2:

The Eigenvalue Method for homogeneous systems

Announcements:

- HW1-4 and A1 due Tues Jun 22
- Syllabus Quiz on Brightspace
- Office Hours Today 2:30-3:30pm on zoom
- Online students can take exams in-person w/ sec 001 → email Dr. Hood

Warm up:

Write the 1st order linear system:

$$x_1' = 2x_1 - 3x_2$$
$$x_2' = -7x_1 + x_2$$

in matrix form.

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}' = \begin{bmatrix} 2 & -3 \\ -7 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

I. Eigenvalue Method:

$$\underline{x}' = \underline{\underline{A}}\,\underline{x}$$

$\underline{\underline{A}}$ is a $n \times n$ constant matrix

Recall:

- 1st order linear ODE (scalar)

$$x'(t) = \lambda x \longrightarrow x(t) = x_0 e^{\lambda t}$$

- 2nd order linear ODE

$$a x'' + b x' + c x = 0$$

assumed solns $x(t) = e^{rt}$

characteristic eqn: $ar^2 + br + c = 0$

roots r_1 and r_2

$$x(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$$

Try something similar

Assume solns: $\underline{x}(t) = e^{\lambda t}\,\underline{v} = e^{\lambda t}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} v_1 e^{\lambda t} \\ v_2 e^{\lambda t} \end{bmatrix}$

Plug into ODE:

$$\ldots = Ax = A(e^{\lambda t}\underline{v}) = \cancel{e^{\lambda t}}(\underline{A}\,\underline{v})$$

Plug into ODE:

$$\underline{x}' = \lambda \cancel{e^{\lambda t}}\, \underline{v} = \underline{\underline{A}}\,\underline{x} = \underline{\underline{A}}\left(e^{\lambda t}\underline{v}\right) = \cancel{e^{\lambda t}}\left(\underline{\underline{A}}\,\underline{v}\right)$$

$$\boxed{\lambda \underline{v} = \underline{\underline{A}}\,\underline{v}}$$ eigenvalue problem

identity matrix

Rewrite:

$$\underline{\underline{A}}\,\underline{v} - \lambda \underline{v} = \underline{0}$$

$$\underline{\underline{A}}\,\underline{v} - \lambda \underline{\underline{I}}\,\underline{v} = \underline{0}$$

$$\underbrace{\boxed{\left(\underline{\underline{A}} - \lambda \underline{\underline{I}}\right)\underline{v} = \underline{0}}}_{\text{solve for } \lambda \text{ and } \underline{v}}$$

$$\underline{\underline{I}} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\underline{\underline{I}}\,\underline{x} = \underline{x}$$

NOTE: This is the vector equivalent of the characteristic equation

λ is called an eigenvalue

$\underline{v}$ is called an eigenvector

Ex: $\underline{x}' = \begin{bmatrix} 0 & 1 \\ 3 & 2 \end{bmatrix} \underline{x}$ solns look like $e^{\lambda t}\underline{v}$

Need to solve $\left(\underline{\underline{A}} - \lambda \underline{\underline{I}}\right)\underline{v} = \underline{0}$

This system has a solution when

$$\det\left(\underline{\underline{A}} - \lambda \underline{\underline{I}}\right) = 0$$

$$\lambda \underline{\underline{I}} = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$$

$$\det \begin{bmatrix} -\lambda & 1 \\ 3 & 2-\lambda \end{bmatrix} = -\lambda(2-\lambda) - 1(3) = 0$$

$$\lambda^2 - 2\lambda - 3 = 0$$

characteristic eqn.

$$(\lambda - 3)(\lambda + 1) = 0$$

$$\boxed{\begin{array}{c} \lambda = 3, -1 \\ \text{eigenvalues} \end{array}}$$

For each eigenvalue there is a corresponding

For each eigenvalue, there is a corresponding eigenvector

$\lambda_1 = 3 \longleftrightarrow \underline{v}^{(1)} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$

To find $\underline{v}^{(1)}$ solve $(\underline{\underline{A}} - \lambda_1 \underline{\underline{I}})\underline{v}^{(1)} = \underline{0}$

$$(\underline{\underline{A}} - 3\underline{\underline{I}})\underline{v}^{(1)} = \underline{0}$$

$$\begin{bmatrix} 0-3 & 1 \\ 3 & 2-3 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

expect to have infinitely many solns

$$\begin{bmatrix} -3 & 1 \\ 3 & -1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \begin{matrix} \to & -3v_1 + v_2 = 0 \\ \to & 3v_1 - v_2 = 0 \end{matrix}$$

one unique equation and two unknowns

v_1 is a free variable, let $v_1 = 1$

$$\underline{v}^{(1)} = \begin{bmatrix} v_1 \\ 3v_1 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

$v_2 = 3v_1$ $\quad$ v_1 is a free variable

$$\boxed{\underbrace{\lambda_1 = 3}_{\text{eigenvalue}} \qquad \underbrace{\underline{v}^{(1)} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}}_{\text{eigenvector}}}$$

$$\underline{\underline{A}}\underline{v} = \lambda\underline{v}$$

$$\underline{\underline{A}}\begin{bmatrix} 1 \\ 3 \end{bmatrix} = 3\begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

$$2\left[3\underline{v}\right] = 2\left[\underline{\underline{A}}\underline{v}\right] = \underline{\underline{A}}\begin{bmatrix} 2 \\ 6 \end{bmatrix} = 3 \cdot \begin{bmatrix} 2 \\ 6 \end{bmatrix}$$

a constant multiple of $\underline{v}$ is also an eigenvector

$$\underline{\underline{A}}(k\underline{v}) = k(\underline{\underline{A}}\underline{v}) = k(\lambda\underline{v}) = \lambda(k\underline{v})$$

Fundamental solution $\underline{x}^{(1)}(t) = e^{3t}\begin{bmatrix} 1 \\ 3 \end{bmatrix} = e^{\lambda_1 t}\underline{v}^{(1)}$

Check that $\underline{x}^{(1)}$ solves $\underline{x}' = \underline{\underline{A}}\underline{x}$

Find eigenvector for $\lambda_2 = -1$ $\quad$ Find $\underline{v}^{(2)} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$

Find eigenvector for $\lambda_2 = -1$ Find $\underline{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$

solve: $(\underline{\underline{A}} - \lambda_2 \underline{\underline{I}})\underline{v}^{(2)} = \underline{0}$ $\underline{\underline{A}} = \begin{bmatrix} 0 & 1 \\ 3 & 2 \end{bmatrix}$

$$\begin{bmatrix} 1 & 1 \\ 3 & 3 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$v_1 + v_2 = 0$

$3v_1 + 3v_2 = 0$

one unique eqn
2 unknowns

$v_2 = -v_1$

here v_1 is a free variable

$$\underline{v}^{(2)} = \begin{bmatrix} v_1 \\ -v_1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Fundamental solution $\underline{x}^{(2)} = e^{\lambda_2 t}\underline{v}^{(2)} = e^{-t}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

Use the Principle of Superposition to find the general soln:

$$\underline{x}(t) = c_1 \underline{x}^{(1)}(t) + c_2 \underline{x}^{(2)}(t)$$

$$\boxed{\underline{x}(t) = c_1 e^{3t}\begin{bmatrix} 1 \\ 3 \end{bmatrix} + c_2 e^{-t}\begin{bmatrix} 1 \\ -1 \end{bmatrix}}$$

II. Graphical Interpretation:

eigenvalues

$\lambda_1 = 3$

$\lambda_2 = -1$

eigenvectors

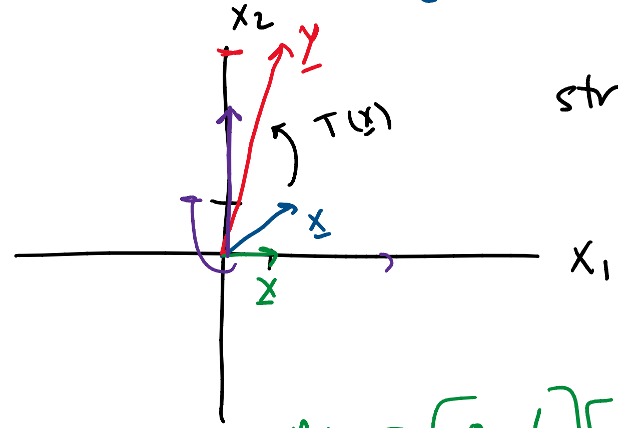
$\underline{v}^{(1)} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

$\underline{v}^{(2)} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

Think of $T(\underline{x}) = \underline{\underline{A}}\,\underline{x}$ as transformation

$$\underline{x} \longrightarrow \boxed{T(\underline{x})} \longrightarrow \underline{y} = \underline{\underline{A}}\,\underline{x}$$

$\underline{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\quad \underline{y} = \underline{\underline{A}}\,\underline{x} = \begin{bmatrix} 0 & 1 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$



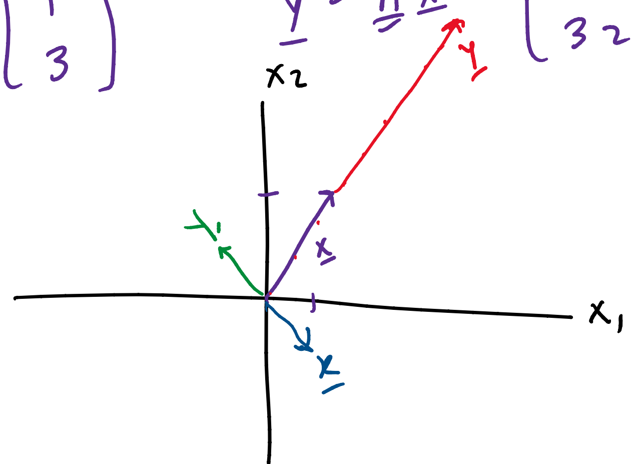
stretches and rotates the vector $\underline{x}$

$\underline{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\quad \underline{y} = \underline{\underline{A}}\,\underline{x} = \begin{bmatrix} 0 & 1 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$

$T(\underline{x})$ stretches + rotates $\underline{x}$

Q: what happens if $\underline{x} = \underline{v}^{(1)}$ eigenvector?

$\underline{x} = \underline{v}^{(1)} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ $\quad \underline{y} = \underline{\underline{A}}\,\underline{x} = \begin{bmatrix} 0 & 1 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 3 \end{bmatrix} = 3\begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 9 \end{bmatrix}$



$\underline{\underline{A}}\,\underline{v}^{(1)} = 3\,\underline{v}^{(1)}$

T stretches by +3

no rotation

$\underline{x} = \underline{v}^{(2)} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ $\quad \underline{y} = \underline{\underline{A}}\,\underline{x} = -1\,\underline{v}^{(2)} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

Edit: $\underline{\underline{A}}(e^{\lambda_1 t}\underline{v}^{(1)}) = e^{\lambda_1 t}(\lambda_1 \underline{v}^{(1)})$

In terms of the ODE $\quad \dfrac{d\underline{x}^{(1)}}{dt} = \underline{\underline{A}}\,\underline{x}^{(1)} = \lambda_1 \underline{x}^{(1)}$

$\underline{x}^{(1)}(t) = e^{3t}\begin{bmatrix} 1 \\ \ldots \end{bmatrix}$

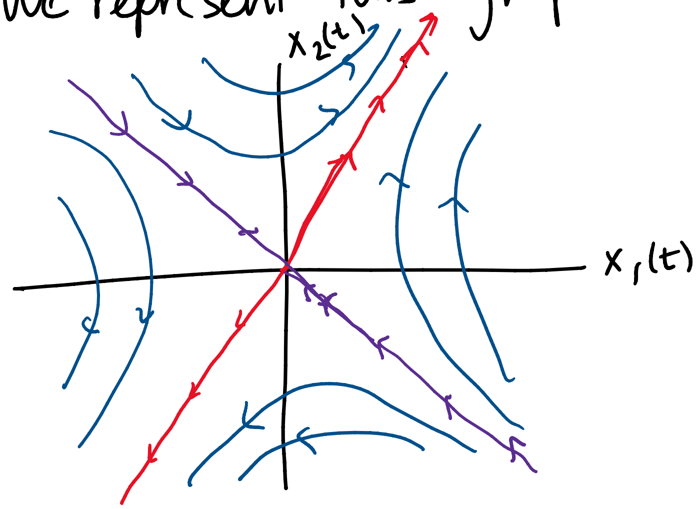
curves grow

$$x^{(1)}(t) = e^{3t}\begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

$\frac{dx}{dt}$

solution curves grow exponentially along $v^{(1)}$

We represent this graphically in a <u>phase portrait</u>



1. Draw eigenvectors
2. If λ is ⊕ grow
   λ is ⊖ decay
3. draw outside curves

$$\frac{d x^{(2)}}{dt} = A x^{(2)} = \lambda_2 x^{(2)}$$

solution curves decay exponentially to origin along $v^{(2)}$

$\boxed{\text{Saddle point}}$

$x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$ plotting $x_1(t)$ vs $x_2(t)$

III. Eigenvalue Method:

Given $x' = A x$ (A is $n \times n$ matrix)

0. Guess $x = e^{\lambda t} v$, plug into ODE to obtain
   $$A v = \lambda v$$
1. Find n eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ by solving $\det(A - \lambda I) = 0$
2. For each λ_i find the corresponding eigenvector $v^{(i)}$
   solve $(A - \lambda_i I) v^{(i)} = 0$

3. General solution is:

$$\underline{x}(t) = c_1 e^{\lambda_1 t} \underline{v}^{(1)} + c_2 e^{\lambda_2 t} \underline{v}^{(2)} + \dots + c_n e^{\lambda_n t} \underline{v}^{(n)}$$

4. Plug in initial condition $\underline{x}(0) = \underline{x}_0$
solve for $c_1, \dots, c_n$

$$\det(\underline{\underline{A}} - \lambda \underline{\underline{I}}) = 0$$

$$a\lambda^2 + b\lambda + c = 0$$

roots $\lambda = a \pm bi$

$\underline{\underline{A}}$ is 2×2

$\underline{Q}$: What happens is λ are complex valued

$\underline{\text{IV. Complex Eigenvalues:}}$

$\underline{\text{Ex}}$: $\underline{x}' = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \underline{x}$

1. eigenvalues $\det(\underline{\underline{A}} - \lambda \underline{\underline{I}}) = 0$

$$\begin{vmatrix} 1-\lambda & 1 \\ -1 & 1-\lambda \end{vmatrix} = (1-\lambda)^2 + 1 = 0$$

$$\sqrt{(1-\lambda)^2} = \sqrt{-1}$$

$$1 - \lambda = \pm i$$

$$\boxed{\lambda = 1 \pm i}$$

$\underline{\text{NOTE}}$: complex eigenvalues always appear in conjugate pairs

2. eigenvectors:

$\boxed{\lambda_1 = 1 + i}$

$$(\underline{\underline{A}} - \lambda \underline{\underline{I}}) \underline{v} = \underline{0}$$

$$\begin{bmatrix} 1-(1+i) & 1 \\ -1 & 1-(1+i) \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -i & 1 \\ \cdot & \cdot \end{bmatrix} \begin{bmatrix} v_1 \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ \cdot \end{bmatrix} \rightarrow -i v_1 + v_2 = 0$$

$$v_2 = i v_1$$

$$\begin{bmatrix} -i & 1 \\ -1 & -i \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightarrow -i v_1 + v_2 = 0$$

$$v_2 = i v_1$$

$$-v_1 - i v_2 = 0$$

$$-v_1 = i v_2$$

$$i v_1 = (-i)(-v_1) = -\frac{v_1}{i} = v_2$$

$$\left(\frac{1}{i} = -i \right)$$

v_1 is a free variable, choose $v_1 = 1$

$$\underline{v}^{(1)} = \begin{bmatrix} v_1 \\ i v_1 \end{bmatrix} = \begin{bmatrix} 1 \\ i \end{bmatrix}$$

$$\boxed{\lambda_2 = 1 - i}$$

$$(\underline{\underline{A}} - \lambda_2 \underline{\underline{I}})\, \underline{v}^{(2)} = \underline{0}$$

$$\underline{v}^{(2)} = \begin{bmatrix} 1 \\ -i \end{bmatrix}$$

eigenvectors are conjugate pairs

Fundamental solutions:

$$\underline{x}^{(1)} = e^{(1+i)t} \begin{bmatrix} 1 \\ i \end{bmatrix}, \quad \underline{x}^{(2)} = e^{(1-i)t} \begin{bmatrix} 1 \\ -i \end{bmatrix}$$

one general solution

$$\underline{x}(t) = c_1 e^{(1+i)t} \begin{bmatrix} 1 \\ i \end{bmatrix} + c_2 e^{(1-i)t} \begin{bmatrix} 1 \\ -i \end{bmatrix}$$

WANT: Real-valued solution

Find a set of fundamental solns $\hat{\underline{x}}^{(1)}$ and $\hat{\underline{x}}^{(2)}$ that are real-valued.

Euler's formula $\quad e^{it} = \cos(t) + i \sin(t)$

$$\underline{x}^{(1)} = e^{t} e^{it} \begin{bmatrix} 1 \\ i \end{bmatrix} = e^{t} \left(\cos(t) + i \sin(t) \right) \begin{bmatrix} 1 \\ i \end{bmatrix}$$

$$e^{t} \Big[\cos(t) \quad + i \sin(t) \Big]$$

$$= e^t \begin{bmatrix} \cos(t) + i\sin(t) \\ i\cos(t) - \sin(t) \end{bmatrix}$$

$$= e^t \left\{ \underbrace{\begin{bmatrix} \cos(t) \\ -\sin(t) \end{bmatrix}}_{\underline{w}} + i \underbrace{\begin{bmatrix} \sin(t) \\ \cos(t) \end{bmatrix}}_{\underline{v}} \right\} = e^t(\underline{w} + i\underline{v})$$

Since $\underline{x}^{(2)}$ is complex conj of $\underline{x}^{(1)}$

$$\underline{x}^{(2)} = e^t(\underline{w} - i\underline{v})$$

New basis ~~for~~ fundamental solns

$$\hat{\underline{x}}^{(1)} = \frac{1}{2}(\underline{x}^{(1)} + \underline{x}^{(2)}) = \frac{1}{2}\left[e^t\{\underline{w} + \cancel{i\underline{v}}\} + e^t\{\underline{w} - \cancel{i\underline{v}}\}\right]$$

$$\boxed{\hat{\underline{x}}^{(1)} = e^t \underline{w}}$$

$$\hat{\underline{x}}^{(2)} = \frac{1}{2i}(\underline{x}^{(1)} - \underline{x}^{(2)}) = e^t \underline{v}$$

Real-valued solution

$$\boxed{\underline{x}(t) = c_1 e^t \underline{w} + c_2 e^t \underline{v}}$$

$$\boxed{= c_1 e^t \begin{bmatrix} \cos(t) \\ -\sin(t) \end{bmatrix} + c_2 e^t \begin{bmatrix} \sin(t) \\ \cos(t) \end{bmatrix}}$$

Next time: phase portrait