Lecture 30

Section 7.5: Piecewise Continuous Input Functions

Announcements:
- Online HW + A7 due Tues Aug 3
- Midterm 2 grades posted
- Final Exam Friday Aug 6 8am-10am
- Evening Thurs Aug 5 6pm-8pm

Warm up:

Recall the unit step function is defined as:

$$u(t-a) = \begin{cases} 0 & t<a \\ 1 & t\geq a \end{cases}$$

Sec 7.1 $\mathcal{L}\{u(t-a)\} = \frac{e^{-as}}{s}$

What is $\mathcal{L}\{u(t-3)\} = ? = \frac{e^{-3s}}{s}$

(a) $\frac{3e^{-s}}{s}$ (b) e^{-3s} (c) $\boxed{\frac{e^{-3s}}{s}}$ (d) $\frac{3}{s}$ (e) $\frac{e^{3s}}{s}$

We've solving ODEs using L.T.

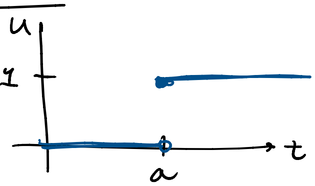
$$ax'' + bx' + cx = f(t)$$

One advantage of the L.T. is we can solve ODEs when $f(t)$ is piecewise continuous

I. Piecewise Continuous Functions

unit step function:

$$u(t-a) = u_a(t) = \begin{cases} 0 & t<a \\ 1 & t\geq a \end{cases}$$



"flipping a switch @ $t=a$"

$$\mathcal{L}\{u(t-a)\} = \frac{e^{-as}}{s}$$

We can also think of it as

$$\mathcal{L}\{u(t-a)(1)\} = e^{-as}\mathcal{L}\{1\} = e^{-as}\left(\frac{1}{s}\right)$$

Thm: (Translation on the t-axis)

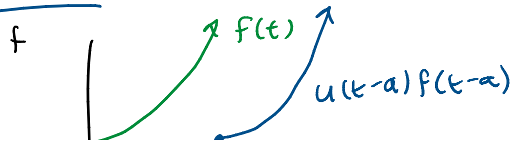
$$\mathcal{L}\{u(t-a)f(t-a)\} = e^{-as}F(s)$$

and conversely

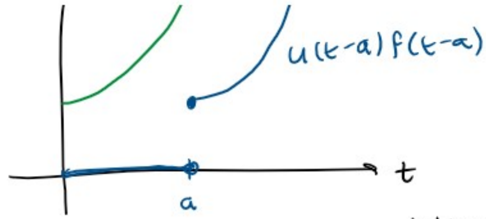
$$\mathcal{L}^{-1}\{e^{as}F(s)\} = u(t-a)f(t-a)$$

t	s
translate by $-a$	multiply by e^{-as}

NOTE: $u(t-a)f(t-a) = \begin{cases} 0 & t<a \\ f(t-a) & t\geq a \end{cases}$



@ $t=a$



@ $t=a$

$u(t-a)f(t-a)$

$= f(t-a) = f(0)$

"time delay of a"

Ex: Find $\mathcal{L}^{-1}\left\{\frac{e^{-5s}}{s^3}\right\} = \mathcal{L}^{-1}\left\{e^{-5s}\underbrace{\left(\frac{1}{s^3}\right)}_{F(s)}\right\}$

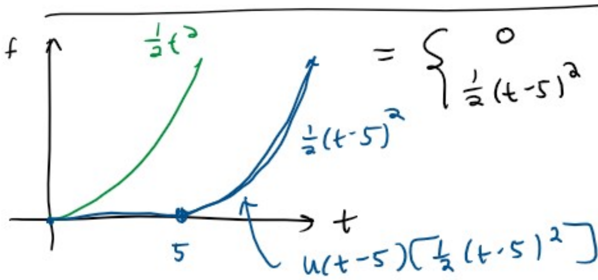
Thm $= u(t-5)f(t-5)$

$F(s) = \frac{1}{s^3}$ then $f(t) = \mathcal{L}^{-1}\left\{\frac{1}{s^3}\right\} =$

$= \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{2}{s^3}\right\} = \frac{1}{2}t^2$

$f(t-5) = \frac{1}{2}(t-5)^2$

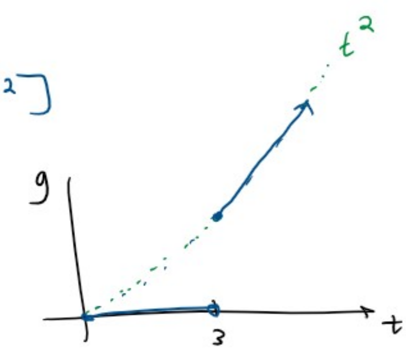
$$\boxed{\mathcal{L}^{-1}\left\{\frac{e^{-5s}}{s^3}\right\} = u(t-5)\left[\frac{1}{2}(t-5)^2\right]}$$



$$= \begin{cases} 0 & t<5 \\ \frac{1}{2}(t-5)^2 & t\geq 5 \end{cases}$$

Ex: Find $\mathcal{L}\{g(t)\}$ where

$$g(t) = \begin{cases} 0 & t<3 \\ t^2 & t\geq 3 \end{cases}$$



$g(t) = t^2 u(t-3)$

Want to use $\mathcal{L}\{f(t-3)u(t-3)\} = e^{-3s}F(s)$

Need to write t^2 in the form $f(t-3)$

$f(t-3) = t^2$

Let $\tau = t-3$ $\quad t = \tau+3$

$f(\tau) = t^2 = (\tau+3)^2 = \tau^2 + 6\tau + 9$

$f(t-3) = ((t-3)+3)^2 = t^2$

Another example

$g(t) = t\,u(t-3)$

$= f(t-3)\,u(t-3)$

$= (t-3+3)\,u(t-3)$

$f(t) = t+3$

$f(t-3) = t$

$f(t-3) = ((t-3)+3)^2 = t^2$

$$\mathcal{L}\{t^2 u(t-3)\} = \mathcal{L}\{f(t-3)u(t-3)\}$$
$$= e^{-3s} F(s)$$

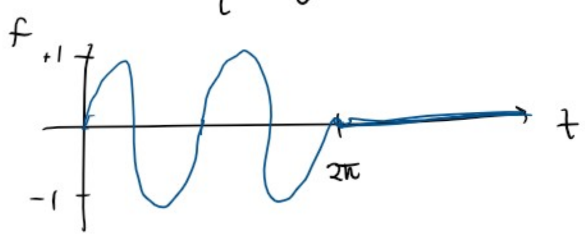
$f(t) = t^2 + 6t + 9$ $\quad F(s) = \mathcal{L}\{t^2 + 6t + 9\}$
$$= \frac{2!}{s^3} + 6\left(\frac{1}{s^2}\right) + 9\left(\frac{1}{s}\right)$$

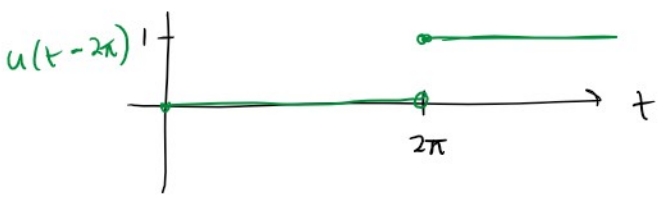
$$\boxed{\mathcal{L}\{t^2 u(t-3)\} = e^{-3s}\left[\frac{2}{s^3} + \frac{6}{s^2} + \frac{9}{s}\right]}$$

$f(t-3) = t$

$g(t) = (t-3)u(t-3)$
$+ 3u(t-3)$

Ex: Find the L.T. of
$$f(t) = \begin{cases} \sin(2t) & 0 \le t < 2\pi \\ 0 & t \ge 2\pi \end{cases}$$





Want the opposite of this
$w(t-2\pi)$

$$w(t-2\pi) = \begin{cases} 1 & t < 2\pi \\ 0 & t \ge 2\pi \end{cases}$$

$f(t) = \sin(2t)w(t-2\pi)$

Let: $\boxed{w(t-2\pi) = 1 - u(t-2\pi)}$
$$= 1 - \begin{cases} 0 & t < 2\pi \\ 1 & t \ge 2\pi \end{cases}$$
$$= \begin{cases} 1-0 & t < 2\pi \\ \cancel{1-1} \to 0 & t \ge 2\pi \end{cases} = \begin{cases} 1 & t < 2\pi \\ 0 & t > 2\pi \end{cases}$$

so we can write
$$f(t) = \sin(2t)\left[1 - u(t-2\pi)\right]$$
$$= \sin(2t) - \underbrace{\sin(2t)\,u(t-2\pi)}_{f(t-2\pi)\,u(t-2\pi)}$$

but $\sin(2t)$ has period π

$$\underbrace{\qquad}_{f(t-2\pi)\,u(t-2\pi)}$$

but $\sin(2t)$ has period π

$\sin(2t) = \sin(2t-4\pi)$

$$= \sin(2t) - \sin(2(t-2\pi))\,u(t-2\pi)$$

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{\sin(2t)\} - \mathcal{L}\{\sin(2(t-2\pi))u(t-2\pi)\}$$

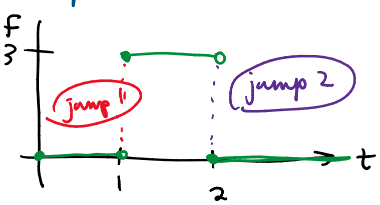
$$= \frac{2}{s^2+4} - e^{-2\pi s}\,\mathcal{L}\{\sin(2t)\}$$

$$= \frac{2}{s^2+4} - e^{-2\pi s}\left(\frac{2}{s^2+4}\right)$$

$$= \boxed{\frac{2\left(1-e^{-2\pi s}\right)}{s^2+4}}$$

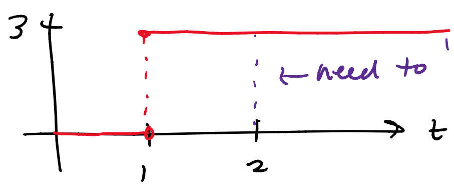
★ Claim: We can write any piecewise continuous function in terms of unit step functions.

Ex: $f(t) = \begin{cases} 0 & t<1 \\ 3 & 1\le t<2 \\ 0 & t\ge 2 \end{cases}$



Two jumps $\longrightarrow$ want to write $f(t)$ in terms $u(t-1)$ and $u(t-2)$

jump 1 is represented by $3u(t-1) = 3\begin{cases} 0 & t<1 \\ 1 & t\ge 1 \end{cases}$



← need to "switch off" at $t=2$

To find jump 2

$$3u(t-1) - f(t) = \begin{cases} 0 & t<1 \\ 3 & 1\le t<2 \\ 3 & t\ge 2 \end{cases} - \begin{cases} 0 & t<1 \\ 3 & 1\le t<2 \\ 0 & t\ge 2 \end{cases}$$

$$= \begin{cases} 0-0 & t<1 \\ \cancel{3-3}\ 0 & 1\le t<2 \\ 3-0 & t\ge 2 \end{cases} = \begin{cases} 0 & t<1 \\ 0 & 1\le t<2 \\ 3 & t\ge 2 \end{cases}$$

$$= \begin{cases} 0 & t<2 \\ 3 & t\ge 2 \end{cases} = 3u(t-2)$$

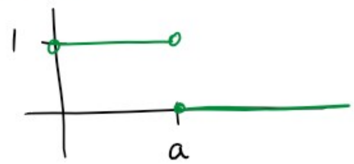
$$\boxed{3u(t-1) - 3u(t-2) = f(t)}$$

Then

$$\mathcal{L}\{f(t)\} = 3\mathcal{L}\{u(t-1)\} - 3\mathcal{L}\{u(t-2)\}$$

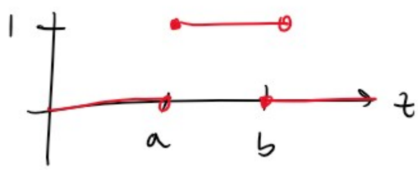
$$= \frac{3e^{-s}}{s} - \frac{3e^{-2s}}{s} = \boxed{\frac{3}{s}\left[e^{-s} - e^{-2s}\right]}$$

NOTE: In the last two examples, we have derived 2 useful piecewise fcns



$$w(t-a) = \begin{cases} 1 & t < a \\ 0 & t \geq a \end{cases}$$

$$\boxed{w(t-a) = 1 - u(t-a)}$$



$$v(t) = \begin{cases} 0 & t < a \\ 1 & a \leq t < b \\ 0 & t \geq b \end{cases}$$

$$\boxed{v(t) = u(t-a) - u(t-b)}$$

From this we can write out any piecewise continuous function in terms of $u(t-a)$

Ex: $f(t) = \begin{cases} 1 & 0 < t < a \\ 2 & a \leq t < b \\ 3 & t \geq b \end{cases}$

$$= 1\underbrace{\begin{cases} 1 & 0 < t < a \\ 0 & a \leq t < b \\ 0 & t \geq b \end{cases}}_{w(t-a) = 1 - u(t-a)} + 2\underbrace{\begin{cases} 0 & 0 < t < a \\ 1 & a \leq t < b \\ 0 & t \geq b \end{cases}}_{v(t) = u(t-a) - u(t-b)} + 3\underbrace{\begin{cases} 0 & 0 < t < a \\ 0 & a \leq t < b \\ 1 & t \geq b \end{cases}}_{u(t-b)}$$

$$= 1\left[1 - u(t-a)\right] + 2\left[u(t-a) - u(t-b)\right] + 3u(t-b)$$

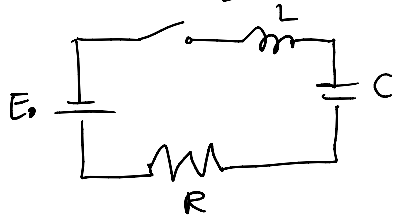
$$\boxed{f(t) = 1 + u(t-a) + u(t-b)}$$

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{1\} + \mathcal{L}\{u(t-a)\} + \mathcal{L}\{u(t-b)\}$$

$$= \frac{1}{s} + \frac{e^{-as}}{s} + \frac{e^{-bs}}{s} = \boxed{\frac{1}{s}\left[1 + e^{-as} + e^{-bs}\right]}$$

III. RLC circuit:

Let $i(t)$ be the current through the circuit



Then:

$$L\frac{di}{dt} + Ri + \frac{1}{C}\int_0^t i(\tau)\,d\tau = e(t)$$

this is called an <u>integrodifferential equation</u>

We can solve this using L.T.

switch closed until $t<1$ switch opened

Ex: $$\frac{di}{dt} + 110i + 1000\int_0^t i(\tau)\,d\tau = 90\left[1-u(t-1)\right]$$

$$i(0)=0$$

Recall: $$\mathcal{L}\left\{\int_0^t i(\tau)\,d\tau\right\} = \frac{I(s)}{s}$$

$$\mathcal{L}\left\{90\left[1-u(t-1)\right]\right\} = \mathcal{L}\{90\} - 90\mathcal{L}\{u(t-1)\}$$

$$= \frac{90}{s} - \frac{90e^{-s}}{s} = \frac{90}{s}\left[1-e^{-s}\right]$$

1. Take the L.T. of both sides

$$\frac{s\left[sI-0\right]}{s} + \frac{110s\left[I\right]}{s} + \frac{1000\,I}{s} = \frac{90}{s}\left[1-e^{-s}\right]$$

$$\frac{\left[s^2+110s+1000\right]}{\cancel{s}}I = \frac{90}{\cancel{s}}\left[1-e^{-s}\right]$$

$$(s+10)(s+100)\,I(s) = 90\left[1-e^{-s}\right]$$

$$I(s) = \frac{90\left[1-e^{-s}\right]}{(s+10)(s+100)}$$

Expand:

$$I(s) = \underbrace{\frac{90}{(s+10)(s+100)}}_{G(s)} - e^{-s}\Bigg(\underbrace{\frac{90}{(s+10)(s+100)}}_{G(s)}\Bigg)$$

$$I(s) = G(s) - e^{-s}G(s)$$

$$i(t) = \mathcal{L}^{-1}\{I(s)\} = g(t) - g(t-1)u(t-1)$$

Need to find $g(t) = \mathcal{L}^{-1}\{G(s)\}$

$$g(t) = \mathcal{L}^{-1}\left\{\frac{90}{(s+10)(s+100)}\right\}$$

Partial fractions

$$\frac{90}{(s+10)(s+100)} = \frac{A}{s+10} + \frac{B}{s+100}$$

.... $A = 1$, $B = -1$

$$g(t) = \mathcal{L}^{-1}\left\{\frac{1}{s+10}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s+100}\right\}$$

$$\boxed{g(t) = e^{-10t} - e^{-100t}}$$

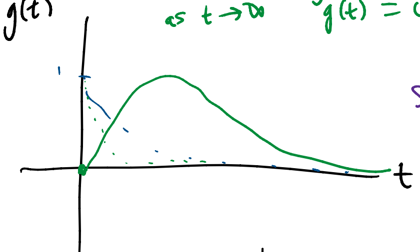
$$i(t) = g(t) - u(t-1)\,g(t-1)$$

$$\boxed{i(t) = \left[e^{-10t} - e^{-100t}\right] - u(t-1)\left[e^{-10(t-1)} - e^{-100(t-1)}\right]}$$

@ $t=0$ $g(t)=0$

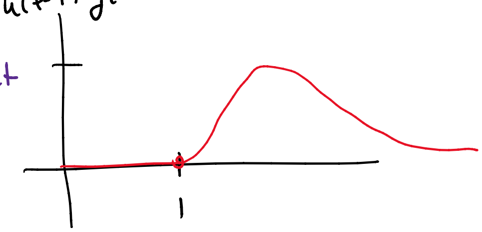
as $t\to\infty$ $g(t)=0$

$g(t)$



subtract

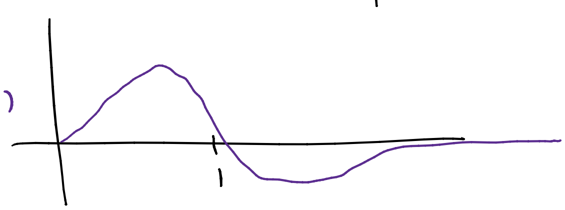
$u(t-1)\,g(t-1)$



@ $t=0$ $i(0)=0$

@ $t=1$ $i(1)=g(1)$

as $t\to\infty$ $i(t)\to 0$



☆ Summary:

- Thm: (Translation on t-axis)

$$\mathcal{L}\{u(t-a)f(t-a)\} = e^{-as}F(s)$$

$$\mathcal{L}^{-1}\{e^{-as}F(s)\} = f(t-a)u(t-a)$$

- $w(t-a) = 1 - u(t-a) = \begin{cases} 1 & t<a \\ 0 & t\geq a \end{cases}$

- $v(t) = u(t-a) - u(t-b) = \begin{cases} 0 & 0<t<a \\ 1 & a\leq t<b \end{cases}$

- $V(t) = u(t-a) - u(t-b) = \left\{ \begin{array}{ll} 0 & 0 < t < a \\ 1 & a \leq t < b \\ 0 & t \geq b \end{array} \right\}$

- any piecewise continuous function can be written in terms of unit step functions.