Lecture 31

Section 7.6: Impulses & Delta Functions

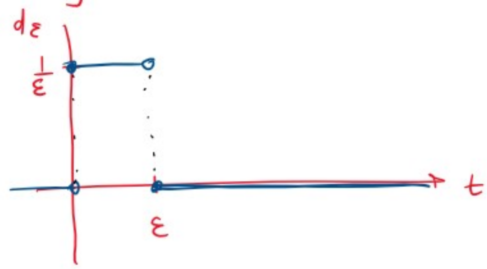
Announcements:
Online HW + A7 due Tues 8/3
Final Exam Friday Aug 6 8am–10am
Evening Exam Thur Aug 5 6pm–8pm
Review in Class Tomorrow

Warm up:
Plot the piecewise continuous function

$$d_\varepsilon(t) = \frac{1}{\varepsilon}\left[u(t) - u(t-\varepsilon)\right] \qquad \text{when } 0<\varepsilon<1$$

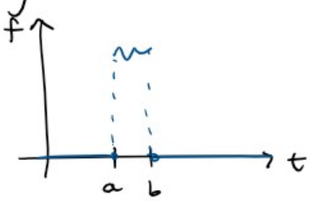
@ $t=0$ positive jump $\frac{1}{\varepsilon}$

@ $t=\varepsilon$ negative jump $\frac{1}{\varepsilon}$



I. Delta Function:

Motivation:
- impulsive force – acts on a short time
  e.g.: base ball bat striking a ball
  surge voltage



The effect depends on the impulse p of the function $f(t)$ over the interval $[a,b]$

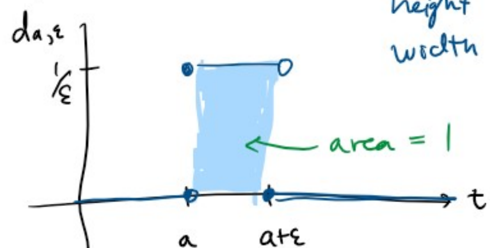
$$p = \int_a^b f(t)\,dt$$

Q: How do we represent this?

Def: The unit impulse $d_{a,\varepsilon}(t)$ is defined

$$d_{a,\varepsilon}(t) = \begin{cases} 1/\varepsilon & \text{if } a \le t < a+\varepsilon \\ 0 & \text{otherwise} \end{cases}$$

height $1/\varepsilon$
width ε



impulse:

$$p = \int_a^{a+\varepsilon} d_{a,\varepsilon}(t)\,dt$$

$$= \int_a^{a+\varepsilon} \frac{1}{\varepsilon}\,dt = \left[\frac{t}{\varepsilon}\right]_a^{a+\varepsilon}$$

$$= \frac{a+\varepsilon}{\varepsilon} - \frac{a}{\varepsilon} = \frac{\varepsilon}{\varepsilon} = \boxed{1 = p}$$

Q: What happens as $\varepsilon \to 0$?

$$\lim d_{a,\varepsilon}(t) = \lim_{\varepsilon\to 0} \left\{ \begin{cases} \frac{1}{\varepsilon} & a \le t < a+\varepsilon \\ 0 & \text{otherwise} \end{cases} \right\}$$

$$\lim_{\varepsilon \to 0} d_{a,\varepsilon}(t) = \lim_{\varepsilon \to 0} \begin{Bmatrix} \frac{1}{\varepsilon} & a \le t < a+\varepsilon \\ 0 & \text{otherwise} \end{Bmatrix}$$

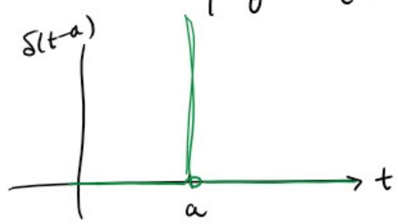
$$= \boxed{\begin{Bmatrix} +\infty & t = a \\ 0 & t \neq a \end{Bmatrix} = \delta(t-a) \quad \text{delta function}}$$

impulse: $p = \int_0^\infty \delta(t-a)\,dt = \lim_{\varepsilon \to 0} \int_0^\infty d_{a,\varepsilon}(t)\,dt$

$$= \lim_{\varepsilon \to 0} [1] = 1$$

Def: The Dirac delta function is $\delta(t-a)$

$$\delta(t-a) = \begin{Bmatrix} +\infty & t=a \\ 0 & t \neq a \end{Bmatrix} \text{ and } \int_0^\infty \delta(t-a)\,dt = 1$$



NOTE: $\delta(t-a)$ is not really a "function", but its a useful concept because it has the following property

$$\int_0^\infty g(t)\,\delta(t-a)\,dt = g(a)$$ ← picks out the value of $g(t)$ at $t=a$

$g(t)$ function $\longrightarrow \boxed{\int_0^\infty g(t)\,\delta(t-a)\,dt} \longrightarrow g(a)$ number

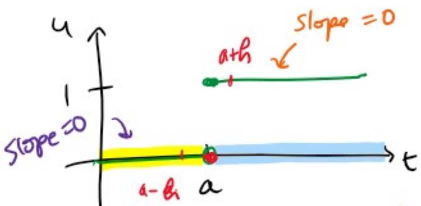
Q: What is $\mathcal{L}\{\delta(t-a)\} = ?$

$$\mathcal{L}\{\delta(t-a)\} = \int_0^\infty e^{-st}\,\delta(t-a)\,dt = \left[e^{-st}\right]\Big|_{t=a}$$

$$\boxed{\mathcal{L}\{\delta(t-a)\} = e^{-as}}$$

NOTE: We can also think of the Dirac delta fcn as the derivative of the unit step function

$$\frac{d}{dt}\left[u(t-a)\right] = \frac{d}{dt}\begin{Bmatrix} 0 & t<a \\ 1 & t \geq a \end{Bmatrix}$$



$$= \begin{cases} 0 & t < a \\ +\infty & t = a \\ 0 & t > a \end{cases} = \delta(t-a)$$

when $t = a$ $\quad \lim_{h \to 0} \dfrac{u(\cancel{a-a}+h) - u(\cancel{a-a}-h)}{2h} = \lim_{h\to 0} \dfrac{u(h) - u(-h)}{2h}$

$$= \lim_{h \to 0} \frac{1-0}{2h} = +\infty$$

$$\mathcal{L}\{\delta(t-a)\} = \mathcal{L}\left\{\frac{d}{dt}\left[u(t-a)\right]\right\}$$

Transform of derivatives property

$$= s\,\mathcal{L}\{u(t-a)\} - \cancel{u(0-a)}^{\,0}$$

$$= \cancel{s}\left[\frac{e^{-as}}{\cancel{s}}\right] = \boxed{e^{-as} = \mathcal{L}\{\delta(t-a)\}}$$

II. IVPs:

GOAL: Solve IVPs w/ forcing terms that are Dirac delta functions

Ex: $x'' + 4x = \delta(t - \pi) \qquad x(0) = x'(0) = 0$

@ $t = \pi$, there is an instantaneous force (impulse)

1. Take the L.T. of both sides

$$\left[s^2 X(s) - s\cdot 0 - 0\right] + 4\left[X(s)\right] = e^{-\pi s}$$

$$(s^2+4)X(s) = e^{-\pi s}$$

$$X(s) = \frac{e^{-\pi s}}{s^2+4}$$

2. Take the inverse L.T.

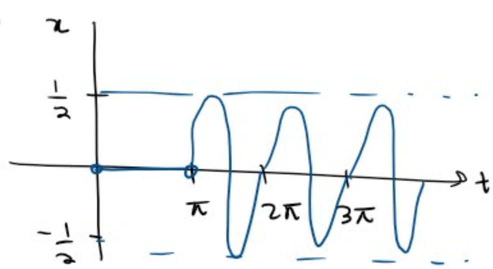
$$x(t) = \mathcal{L}^{-1}\left\{\frac{e^{-\pi s}}{s^2+4}\right\} = \mathcal{L}^{-1}\left\{\frac{e^{-\pi s}}{2}\underbrace{\left(\frac{2}{s^2+4}\right)}_{\mathcal{L}\{\sin(2t)\}}\right\}$$

$$= \frac{1}{2}u(t-\pi)\sin(2(t-\pi))$$

$$= \frac{1}{2}u(t-\pi)\sin(2t - 2\pi) \leftarrow \text{periodic w/ } p = \pi$$

$$\boxed{x(t) = \frac{1}{2}\sin(2t)\,u(t-\pi)}$$

$$= \begin{cases} 0 & t < \pi \\ \frac{1}{2}\sin(2t) & t \geq \pi \end{cases}$$



III. Duhamel's Principle:

Consider a physical system represented by

$$ax'' + bx' + cx = f(t) \qquad x(0) = x'(0) = 0$$

$x(t)$ - output or response

$f(t)$ - input

1. Take the L.T. of both sides

$$a[s^2 X(s) - 0\cdot s - 0] + b[sX - 0] + cX = F(s)$$

$$(as^2 + bs + c)\,X(s) = F(s)$$

$$X(s) = \frac{F(s)}{as^2+bs+c} = F(s)\underbrace{\left(\frac{1}{as^2+bs+c}\right)}_{W(s)}$$

$$X(s) = F(s)W(s)$$

$W(s) = \dfrac{1}{as^2+bs+c}$ transfer function of the system

$w(t) = \mathcal{L}^{-1}\{W(s)\}$ - weight function

2. Take the inverse L.T.

→ use convolution property

$$x(t) = \mathcal{L}^{-1}\{F(s)\,W(s)\} = (f * w)(t)$$

$$\boxed{x(t) = \int_0^t w(\tau)\, f(t-\tau)\, d\tau}$$

NOTE: only if $x(0) = x'(0) = 0$

This is called Duhamel's Principle

Key step to solving the IVP is finding

$$w(t) = \mathcal{L}^{-1}\{W(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{as^2+bs+c}\right\}$$

Ex: Apply Duhamel's Principle to write an integral formula for the soln to the IVP:

$$x'' + 6x' + 10x = f(t) \qquad x(0) = x'(0) = 0$$

1. Take L.T.

$$[s^2X - 0s - 0] + 6[sX - 0] + 10(X) = F(s)$$

$$(s^2 + 6s + 10)\,X(s) = F(s)$$

$[s^2X - 0s - 0] + 6[sX - 0] + 10[X] = F(s)$

$$(s^2 + 6s + 10)\,X(s) = F(s)$$

$$X(s) = F(s)W(s)$$

$$W(s) = \frac{1}{s^2+6s+10} \quad \text{transfer fcn}$$

Weight fcn $w(t) = \mathcal{L}^{-1}\left\{ \frac{1}{s^2+6s+10} \right\}$

$$= \mathcal{L}^{-1}\left\{ \frac{1}{(s+3)^2+1} \right\} = e^{-3t}\sin(t)$$

2. Inverse L.T.

$x(t) = \mathcal{L}^{-1}\{F(s)W(s)\} = (f * w)(t)$

$$\boxed{x(t) = \int_0^t e^{-3\tau}\sin(\tau)\, f(t-\tau)\, d\tau}$$

☆ Summary (Delta Fcns)

- Dirac delta function

$$\delta(t-a) = \begin{cases} +\infty & \text{if } t = a \\ 0 & \text{if } t \neq a \end{cases} \quad \underline{\underline{\text{and}}} \quad \int_0^\infty \delta(t-a)\,dt = 1$$

- $\int_0^\infty g(t)\,\delta(t-a)\,dt = g(a)$

- $\mathcal{L}\{\delta(t-a)\} = e^{-as}$

- Duhamel's Principle

$$x(t) = \int_0^t w(\tau)\, f(t-\tau)\, d\tau$$

where $w(t)$ weight function, $W(s)$ transfer function

We have 6 major Thms for Laplace Transforms

Q: When should we use which Thm?

$\mathcal{L}^{-1}\{\cdot\}$	when to use	Examples
Transform of Integrals $\mathcal{L}^{-1}\{\frac{F(s)}{s}\} = \int^t f(\tau)d\tau$	s in the denominator [can also use partial fractions $\frac{1}{\ldots} = \frac{A}{\ldots} + \frac{B}{\ldots}$]	$\frac{1}{s(s-3)}$, $\frac{1}{\ldots}$

Transform of Integrals — s in the denominator

$$\mathcal{L}^{-1}\left\{\frac{F(s)}{s}\right\} = \int_0^t f(\tau)\,d\tau$$

[can also use partial fractions $\frac{1}{s(s-3)} = \frac{A}{s} + \frac{B}{s-3}$]

$\frac{1}{s(s-3)}$

$\frac{1}{s(s^2+a^2)}$

Translation on the s-axis

$$\mathcal{L}\{e^{at} f(t)\} = F(s-a)$$

$$\mathcal{L}^{-1}\{F(s-a)\} = e^{at} f(t)$$

often use when doing partial fractions

$\frac{1}{(s-a)^2+b^2} = \frac{1}{b} e^{at}\sin(bt)$

$\frac{1}{(s-a)^n} = e^{at}\frac{t^{n-1}}{n!}$

Convolution Property

$$\mathcal{L}^{-1}\{F(s)G(s)\} = (f*g)(t)$$

$$= \int_0^t f(\tau)g(t-\tau)\,dt$$

Duhamel's Principle

partial fractions

[can often be replaced by partial fractions]

$\left[\frac{1}{s^2+b^2}\right]^2 = \left(\frac{1}{s^2+b^2}\right)\left(\frac{1}{s^2+b^2}\right),$

$\frac{1}{(s^2+a^2)(s^2+b^2)} = \left(\frac{1}{s^2+a^2}\right)\left(\frac{1}{s^2+b^2}\right),$

$= \frac{As+B}{s^2+a^2} + \frac{Cs+D}{s^2+b^2}$

Differentiation of Transforms

$$\mathcal{L}^{-1}\{F(s)\} = -\frac{1}{t}\mathcal{L}^{-1}\{F'(s)\}$$

use when $F'(s)$ is nicer than $F(s)$

$F(s)$ has a $\ln$ term

$\ln\left(\frac{s-2}{s+2}\right)$

Translation on the t-axis

$$\mathcal{L}^{-1}\{e^{-as}F(s)\} = u(t-a)f(t-a)$$

whenever e^{-as} appears

often $f(t)$ has
$u(t-a)$ term
$\delta(t-a)$ term

$e^{-as}\left(\frac{1}{s^2+b^2}\right)$

$(1-e^{-as})G(s)$

expand

$G(s) - \underbrace{e^{-as}G(s)}_{\text{apply}}$

Ex: Convolution

$$\mathcal{L}^{-1}\left\{\frac{1}{(s-3)^2}\right\} = \mathcal{L}^{-1}\left\{\underbrace{\left(\frac{1}{s-3}\right)}_{F(s)}\underbrace{\left(\frac{1}{s-3}\right)}_{G(s)}\right\}$$

$$f(t) = \mathcal{L}^{-1}\{F(s)\} = e^{3t}$$

$$g(t) = \mathcal{L}^{-1}\left\{\frac{1}{s-3}\right\} = e^{3t}$$

convolution $= (f*g)(t) = \int_0^t f(\tau)g(t-\tau)\,d\tau$

$$= \int_0^t e^{3\tau} e^{3(t-\tau)}\,d\tau$$

$$= \int_0^t e^{\cancel{3\tau}+3t-\cancel{3\tau}}\,d\tau$$

$$= \int_0^t e^{\cancel{3\tau}+3t-\cancel{3\tau}}\, d\tau$$

$$= e^{3t}\int_0^t d\tau = e^{3t}\left[\tau\right]_0^t$$

$$= e^{3t}(t-0) = \boxed{te^{3t} = \mathcal{L}^{-1}\left\{\frac{1}{(s-3)^2}\right\}}$$

$\mathcal{L}\{\cdot\} = \int_0^t$

Translation on t-axis:

$$\mathcal{L}^{-1}\{e^{-as}F(s)\} = u(t-a)f(t-a) \leftarrow a>0$$

$$\mathcal{L}^{-1}\left\{e^{-2s}\underbrace{\left(\frac{s+3}{s^2+1}\right)}_{F(s)}\right\} = u(t-2)f(t-2)$$

$$\mathcal{L}\left\{e^{2s}\left(\frac{s+3}{s^2+1}\right)\right\}$$

$$= u(t+2)f(t+2)$$

assume $a>0$

$$f(t) = \mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\left\{\frac{s+3}{s^2+1}\right\}$$

linearity $= \mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\} + 3\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\}$

$$= \cos(t) + 3\sin(t)$$

$$\mathcal{L}^{-1}\{\cdot\} = u(t-2)\, f(t-2)$$

$$= u(t-2)\left[\cos(t-2) + 3\sin(t-2)\right]$$

Translation on t-axis

$$\mathcal{L}\{e^{at}f(t)\} = F(s-3)$$

on s-axis

$$\mathcal{L}\{f(t-a)u(t-a)\} = e^{-as}F(s)$$