Lecture 4

☆ Section 5.5 - Part 2

Multiple Eigenvalues

Announcements:
- HW 1-4 + A1 due Tues June 22
- Syllabus Quiz
- (Online) Nominate Proctor Survey

Warm up:

Find the eigenvalues of the matrix

$$\underline{\underline{A}} = \begin{bmatrix} 3 & 1 & 2 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{bmatrix}$$

Ans: $\det(\underline{\underline{A}} - \lambda \underline{\underline{I}}) = 0 = \begin{vmatrix} 3-\lambda & 1 & 2 \\ 0 & 3-\lambda & 1 \\ 0 & 0 & 3-\lambda \end{vmatrix}$ expansion by minors

$$= (3-\lambda)\begin{vmatrix} 3-\lambda & 1 \\ 0 & 3-\lambda \end{vmatrix} - (1)\cancel{\begin{vmatrix} 0 & 1 \\ 0 & 3-\lambda \end{vmatrix}}^{\,0} + (2)\cancel{\begin{vmatrix} 0 & 3-\lambda \\ 0 & 0 \end{vmatrix}}^{\,0}$$

$$= (3-\lambda)\left[(3-\lambda)^2 - 0\right] = (3-\lambda)^3 = 0$$

$$\lambda = 3$$

algebraic mult $k = 3$

I. Multiple Eigenvalues:

p geometric multiplicity (# of eigenvectors $\underline{v}$)

k algebraic multiplicity (# of times λ is repeated)

If $p < k$, then λ is defective

... (one or more) generalized

Need to find (one or more) generalized eigenvectors

Ex: $\begin{bmatrix} 3 & 1 & 2 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{bmatrix}$ $\lambda = 3$, $k = 3$

Look for eigenvectors:

$(\underline{\underline{A}} - 3\underline{\underline{I}})\underline{v} = \underline{0}$

$$\begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \rightarrow \begin{array}{l} v_2 + 2v_3 = 0 \\ v_3 = 0 \rightarrow v_2 = 0 \end{array}$$

no constraints on v_1

v_1 is a free variable

$$\underline{v} = \begin{bmatrix} v_1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$ geometric mult $p = 1$

Here $p = 1 < k = 3$

so $\lambda = 3$ is defective

defect $d = k - p = 3 - 1 = 2$

$\boxed{d = 2}$ Need to find 2 generalized eigenvectors

NOTE: To find a general solution, need $n = 3$ linearly independent solutions

To find more than one generalized eigenvector, To find more than one generalized eigenvector, we need to calculate a <u>chain of generalized eigenvectors</u>

NOTE: In 2D, our defect is at most $d=1$ so we can use $(\underline{A}-\lambda\underline{\underline{I}})\underline{u}=\underline{v}$ to calculate $\underline{u}$

In 3D, we need to do something different

II. Chain of Gen. EV.

1. Calculate $(\underline{A}-\lambda\underline{\underline{I}})^2$ and $(\underline{A}-\lambda\underline{\underline{I}})^3$

   (Need $(\underline{A}-\lambda\underline{\underline{I}})^{d+1}$)

$$(\underline{A}-3\underline{\underline{I}})^2=\begin{bmatrix}0&1&2\\0&0&1\\0&0&0\end{bmatrix}\begin{bmatrix}0&1&2\\0&0&1\\0&0&0\end{bmatrix}=\begin{bmatrix}0&0&1\\0&0&0\\0&0&0\end{bmatrix}$$

$$(\underline{A}-3\underline{\underline{I}})^3=\begin{bmatrix}0&1&2\\0&0&1\\0&0&0\end{bmatrix}\begin{bmatrix}0&0&1\\0&0&0\\0&0&0\end{bmatrix}=\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}$$

2. Find a nonzero vector $\underline{u}^{(3)}$ such that

$$(\underline{A}-3\underline{\underline{I}})^3\underline{u}^{(3)}=\underline{0}$$

$$\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}\begin{bmatrix}u_1\\u_2\\u_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

3 free variables

want $\underline{u}^{(3)}\neq 0$

want $\underline{u}^{(3)}$ and $\underline{v}=\begin{bmatrix}1\\0\\0\end{bmatrix}$ to be linearly indep.

Choose

(could choose

Choose

$$\underline{u}^{(3)} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \qquad \left(\text{could choose } \begin{bmatrix} 200 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 100 \\ -99 \\ 223 \end{bmatrix}, \text{etc.} \right)$$

3. Calculate the chain of generalized eigenvectors

$$\underline{u}^{(2)} = (\underline{\underline{A}} - 3\underline{\underline{I}})\underline{u}^{(3)} = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \boxed{\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} = \underline{u}^{(2)}}$$

$$\underline{u}^{(1)} = (\underline{\underline{A}} - 3\underline{\underline{I}})\underline{u}^{(2)} = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} = \boxed{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \underline{u}^{(1)}}$$

4. Check that $\underline{u}^{(1)}$ is a scalar multiple of the eigenvector $\underline{v}$

$$\underline{u}^{(1)} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad \underline{v} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

yes $\underline{u}^{(1)} = \underline{v}$ $\longrightarrow$ chain is valid

The chain of generalized ev. is:

$$\underline{u}^{(1)} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad \underline{u}^{(2)} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \qquad \underline{u}^{(3)} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

NOTE: Each vector in the chain satisfies

$$\underline{u}^{(i)} = (\underline{\underline{A}} - \lambda\underline{\underline{I}})\underline{u}^{(i+1)}$$

and the chain ends on the eigenvector $\underline{v}$, and the chain ends on the eigenv...

5. Write down fundamental solns:

$$\underline{x}^{(1)} = e^{\lambda t}\underline{u}^{(1)} = e^{3t}\begin{bmatrix}1\\0\\0\end{bmatrix}$$

b/c $\underline{u}^{(1)}$ is an eigenvector

$$\underline{x}^{(2)} = e^{\lambda t}\left\{t\underline{u}^{(1)} + \underline{u}^{(2)}\right\}$$

Same as 2D

$$= e^{3t}\left\{t\begin{bmatrix}1\\0\\0\end{bmatrix} + \begin{bmatrix}2\\1\\0\end{bmatrix}\right\} = e^{3t}\begin{bmatrix}t+2\\1\\0\end{bmatrix}$$

t multiplies the eigenvector

no t dependence on the gen. ev. $\underline{u}^{(2)}$

$$\underline{x}^{(3)} = e^{\lambda t}\left\{\frac{t^2}{2}\underline{u}^{(1)} + t\underline{u}^{(2)} + \underline{u}^{(3)}\right\}$$

$$= e^{3t}\left\{\frac{t^2}{2}\begin{bmatrix}1\\0\\0\end{bmatrix} + t\begin{bmatrix}2\\1\\0\end{bmatrix} + \begin{bmatrix}0\\0\\1\end{bmatrix}\right\} = e^{3t}\begin{bmatrix}\frac{t^2}{2}+2t\\t\\1\end{bmatrix}$$

factor of $\frac{1}{2}$ in front

6. General solution $\longrightarrow$ Superposition

$$\underline{x}(t) = C_1\underline{x}^{(1)} + C_2\underline{x}^{(2)} + C_3\underline{x}^{(3)}$$

$$\boxed{\underline{x}(t) = C_1 e^{3t}\begin{bmatrix}1\\0\\0\end{bmatrix} + C_2 e^{3t}\begin{bmatrix}t+2\\1\\0\end{bmatrix} + C_3 e^{3t}\begin{bmatrix}\frac{t^2}{2}+2t\\t\\1\end{bmatrix}}$$

Q: How many chains

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$n \times n$ matrix $\quad \lambda = 1 \quad k = 5 \quad p = 2$

$\lambda = 2 \quad k = 4 \quad p = 1$

need a chain for $\lambda = 1$ \
need a chain for $\lambda = 2$ } each defective eigenvalue

chain has length $d+1$ $\quad$ start $(\underline{\underline{A}} - \lambda \underline{\underline{I}})^{d+1} \underline{u}^{(d+1)} = \underline{0}$

$\underline{u}^{d}$

$\vdots$

$\underline{u}^{(1)}$

set of eigenvectors + gen. ev.

$$\{\underline{v}^{(1)}, \dots, \underline{v}^{(p)}\} \cup \{\underline{u}^{(1)}, \underline{u}^{(2)}, \dots, \underline{u}^{(d+1)}\} = \underline{V}$$

★ **ALGORITHM**: chain of generalized e.v.

Let λ be a repeated eigenvalue with defect d $\quad (d = k - p)$

0. Find p eigenvalues $\underline{v}^{(1)}, \dots, \underline{v}^{(p)}$
1. Calculate $(\underline{\underline{A}} - \lambda \underline{\underline{I}})^{(d+1)}$
2. Find a nonzero vector $\underline{u}^{(d+1)}$ that satisfies

$$(\underline{\underline{A}} - \lambda \underline{\underline{I}})^{(d+1)} \underline{u}^{(d+1)} = \underline{0}$$

- make sure $\underline{u}^{(d+1)}$ is NOT a multiple of the eigenvectors $\underline{v}^{(1)}, \dots, \underline{v}^{(p)}$

generalized ev. $\underline{u}^{(i)}$ $\quad (i)$

the eigen...

- want generalized ev. $\underline{u}^{(i)}$ to be linearly independent with $\underline{v}^{(i)}$ eigenvectors

3. Calculate the chain
$$\underline{u}^{(i)} = (\underline{\underline{A}} - \lambda \underline{\underline{I}})\underline{u}^{(i+1)} \quad \text{for } i=1,\dots,d$$

4. Check that $\underline{u}^{(1)}$ is a multiple of an eigenvector $\underline{v}$
   * in the span $\{\underline{v}^{(1)},\dots,\underline{v}^{(p)}\}$

5. Fundamental solutions
$$\underline{x}^{(1)} = e^{\lambda t}\underline{u}^{(1)}$$
$$\underline{x}^{(2)} = e^{\lambda t}\left\{ t\underline{u}^{(1)} + \underline{u}^{(2)} \right\}$$
$$\vdots$$
$$\underline{x}^{(i)} = e^{\lambda t}\left\{ \frac{t^i}{i!}\underline{u}^{(1)} + \frac{t^{i-1}}{(i-1)!}\underline{u}^{(2)} + \dots + t\underline{u}^{(i-1)} + \underline{u}^{(i)} \right\}$$

6. Use superposition to get general soln
$$\underline{x}(t) = c_1\underline{x}^{(1)} + \dots + c_k\underline{x}^{(k)}$$

$\underline{v}^{(1)}$ and $\underline{v}^{(2)}$ are eigenvectors with λ and λ

Q: is $\underline{v} = \underline{v}^{(1)} + \underline{v}^{(2)}$ also an ev?

$$\underline{\underline{A}}\,\underline{v} = \underline{\underline{A}}(\underline{v}^{(1)} + \underline{v}^{(2)}) = \underline{\underline{A}}\,\underline{v}^{(1)} + \underline{\underline{A}}\,\underline{v}^{(2)}$$
$$= \lambda\underline{v}^{(1)} + \lambda\underline{v}^{(2)} = \lambda(\underline{v}^{(1)} + \underline{v}^{(2)})$$

$\underline{\underline{A}} = \quad =$

$= \lambda v^{(1)} + \lambda \underline{v}^{(2)} = \lambda (\underline{v}^{(1)} + \underline{v}^{(2)})$

Ex: $\underline{\underline{A}} = \begin{bmatrix} 5 & 1 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$ $\quad \lambda = 5$, alg mult $k = 3$

$\lambda = 5$

$(\underline{\underline{A}} - 5\underline{\underline{I}})\underline{v} = \underline{0}$

$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \rightarrow \quad v_2 = 0$

2 free variables v_1 and v_3

$\underline{v}^{(1)} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ choose $v_1 = 1$, $v_3 = 0$

$\underline{v}^{(2)} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ choose $v_1 = 0$, $v_3 = 1$

Need $\underline{v}^{(1)}$ and $\underline{v}^{(2)}$ to be linearly independent

geometric mult $\boxed{p = 2}$

Defect $d = k - p = 3 - 2 = 1$ $\quad \boxed{d = 1}$

$\lambda = 5$ is defective

Calculate chain of gen. ev.

1. $(\underline{\underline{A}} - 5\underline{\underline{I}})^2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

2. Find $\underline{u}^{(2)}$ st. $(\underline{\underline{A}} - 5\underline{\underline{I}})^2 \underline{u}^{(2)} = \underline{0}$

POLL: Which of the following is a valid choice for $\underline{u}^{(2)}$?

POLL: Which of the following is a valid choice ...

(a) $\begin{bmatrix} 17 \\ 0 \\ 0 \end{bmatrix}$ ✗ (b) $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ ✓ (c) $\begin{bmatrix} 25 \\ 0 \\ -7 \end{bmatrix}$ ✗ (d) $\begin{bmatrix} 0 \\ 999 \\ 0 \end{bmatrix}$ ✓ (e) $\begin{bmatrix} 25 \\ 101 \\ -7 \end{bmatrix}$ ✓

(a) multiple of ev $\underline{v}^{(1)}$

(c) linear comb of $25\underline{v}^{(1)} - 7\underline{v}^{(2)}$ not linearly indep

(e) yes lin. indep with $\underline{v}$

Let's take $\underline{u}^{(2)} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

3. Calculate the chain

$$\underline{u}^{(1)} = (\underline{A} - 5\underline{\underline{I}})\underline{u}^{(2)} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

4. Check that $\underline{u}^{(1)}$ is in the span $\{\underline{v}^{(1)}, \underline{v}^{(2)}\}$

$$\underline{v}^{(1)} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad \underline{v}^{(2)} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

yes ✓ $\underline{u}^{(1)} = \underline{v}^{(1)}$

(If $\underline{u}^{(1)}$ is NOT a multiple of an eigenvector, go back to step 2 and choose a new vector $\underline{u}^{(d+1)}$)

$$\text{chain } \{\underline{u}^{(1)}, \underline{u}^{(2)}\} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$

5. Fundamental solutions

$$\underline{x}^{(1)} = e^{\lambda t}\underline{u}^{(1)} = e^{5t}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\underline{x}^{(2)} = e^{\lambda t}\{t\underline{u}^{(1)} + \underline{u}^{(2)}\} = e^{5t}\left\{ t\begin{bmatrix} 1 \\ 0 \\ \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ \end{bmatrix} \right\}$$

$$\underline{x}^{(2)} = e^{\lambda t}\{t\,\underline{u}^{(1)} + \underline{u}^{(2)}\} = e^{5t}\left\{ t\begin{bmatrix}1\\0\\0\end{bmatrix} + \begin{bmatrix}0\\1\\0\end{bmatrix}\right\}$$

$$= e^{5t}\begin{bmatrix}t\\1\\0\end{bmatrix}$$

$$\underline{x}^{(3)} = e^{\lambda t}\,\underline{v}^{(2)} = e^{5t}\begin{bmatrix}0\\0\\1\end{bmatrix}$$

6. General solution

$$\boxed{\underline{x}(t) = c_1 e^{5t}\begin{bmatrix}1\\0\\0\end{bmatrix} + c_2 e^{5t}\begin{bmatrix}t\\1\\0\end{bmatrix} + c_3 e^{5t}\begin{bmatrix}0\\0\\1\end{bmatrix}}$$

☆ Summary:

- If the defect $d = k - p$ is greater than one ($d \geq 2$)

  then calculate a chain of generalized ev.

- Want the set of all eigenvectors and generalized eigenvectors to be linearly independent.