Lecture 5

☆Section 5.3:

A Gallery of Solution Curves

Warm up:

Find the trace T and determinant D of the matrix

$$\underline{\underline{A}} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Announcements:

HW1-4 + A1 due Tues June 22

Syllabus Quiz on Brightspace

Scope of chapter 5:

only need to solve 2x2 and 3x3 systems of ODE

Ans:

$T = \text{trace}(\underline{\underline{A}}) = a+d$

$D = \det(\underline{\underline{A}}) = ad - bc$

For 2D linear systems:

$$\underline{x}' = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \underline{x}$$

the phase depends on the eigenvalues + eigenvectors of $\underline{\underline{A}}$

eigenvalues $\det(\underline{\underline{A}} - \lambda \underline{\underline{I}}) = 0$

$$\begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} = (a-\lambda)(d-\lambda) - bc = 0$$

$$= \lambda^2 - a\lambda - d\lambda + ad - bc = 0$$

$$\lambda^2 - \underbrace{(a+d)}_{T = \text{trace}(\underline{\underline{A}})}\lambda + \underbrace{(ad-bc)}_{D = \det(\underline{\underline{A}})} = 0$$

$$\boxed{\lambda^2 - T\lambda + D = 0}$$ characteristic eqn.

eigenvalues are roots of 2nd degree polynomial

Three major cases:

I. Real distinct λ

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I. Real distinct λ

II. Repeated λ w/ alg mult $k=2$

III. Complex-valued λ

I Real distinct λ

(a) λ opposite signs

Ex: $\underline{x}' = \begin{bmatrix} 3 & -4 \\ 1 & 2 \end{bmatrix} \underline{x}$

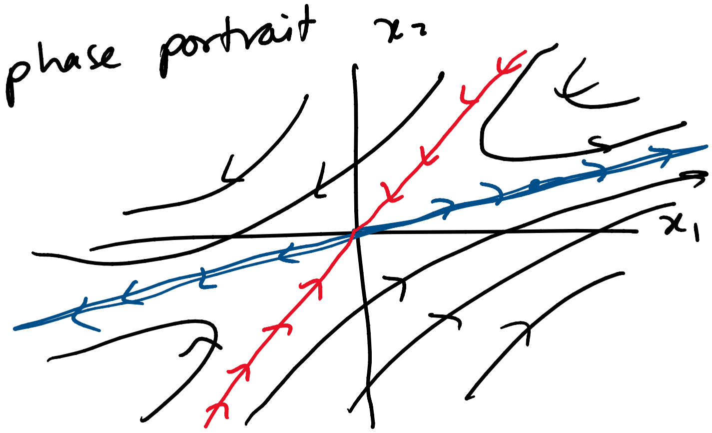
$\lambda_1 = 2$ $\quad \underline{v}^{(1)} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$

$\lambda_2 = -1$ $\quad \underline{v}^{(2)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

general solution:

$$\underline{x}(t) = c_1 e^{2t} \begin{bmatrix} 4 \\ 1 \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

phase portrait



1. Draw eigenvectors
   $\lambda_1 = 2$ $+$
   $\lambda_2 = -1$ $-$
2. Draw connecting curves

$\boxed{\text{saddle point}}$

(b) λ both negative

Ex: $\underline{x}' = \begin{bmatrix} 0 & -1 \\ 2 & -3 \end{bmatrix} \underline{x}$

$\lambda_1 = -1$ $\quad \underline{v}^{(1)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\lambda_2 = -2$ $\quad \underline{v}^{(2)} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

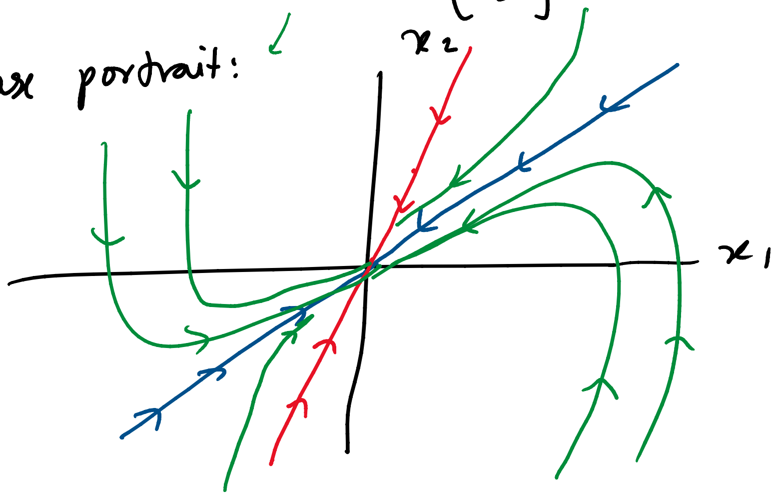
general solution

$$\underline{x}(t) = c_1 e^{-t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{-2t} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

x_2

1. Draw eigenvectors

phase portrait:



1. Draw eigenvectors

$\lambda_1 = -1$ ⊖

$\lambda_2 = -2$ ⊖

2. limit as $t \to \infty$

e^{-t} "dominates"

$\underline{x} \to \underline{0}$ along $\begin{bmatrix}1\\1\end{bmatrix}$

3. limit as $t \to -\infty$

e^{-2t} "dominates"

$\underline{x} \to$ parallel $\begin{bmatrix}1\\2\end{bmatrix}$

$\boxed{\text{improper nodal sink}}$

- multiple solns approach origin
- tangent to $\begin{bmatrix}1\\1\end{bmatrix}$

(c) λ both positive $\quad \underline{x}(t) = c_1 e^{t}\begin{bmatrix}1\\1\end{bmatrix} + c_2 e^{2t}\begin{bmatrix}1\\2\end{bmatrix}$

phase portrait same as (b)

arrows point outward

$\boxed{\text{improper nodal source}}$

(d) one zero eigenvalue

Ex: $\underline{x}' = \begin{bmatrix}1 & -1\\2 & -2\end{bmatrix}\underline{x}$

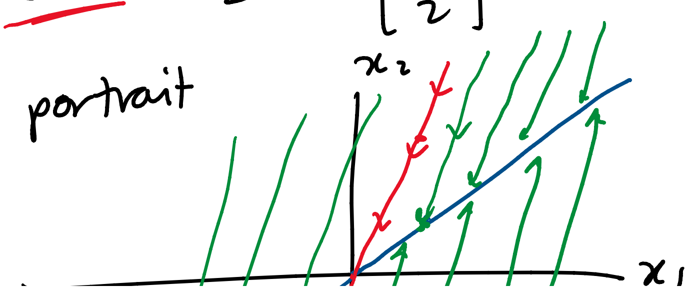
general solution

$\underline{x}(t) = c_1\begin{bmatrix}1\\1\end{bmatrix} + c_2 e^{-t}\begin{bmatrix}1\\2\end{bmatrix}$

$\lambda_1 = 0 \qquad \underline{v}^{(1)} = \begin{bmatrix}1\\1\end{bmatrix}$

$\lambda_2 = -1 \qquad \underline{v}^{(2)} = \begin{bmatrix}1\\2\end{bmatrix}$

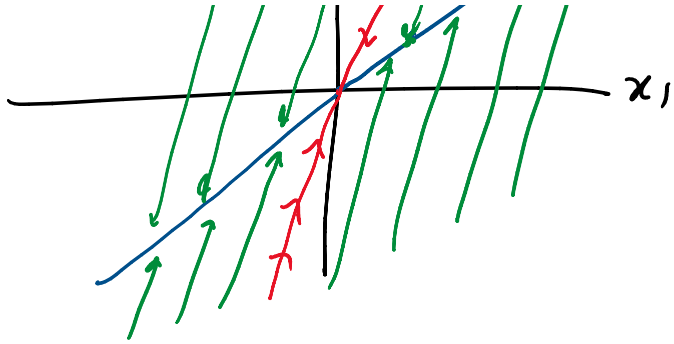
phase portrait



1. Draw eigenvectors

$\lambda_1 = 0$

$\underline{x}^{(1)} = c_1\begin{bmatrix}1\\1\end{bmatrix}$



$\underline{x} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\underline{x}' = \underline{\underline{A}}\, \underline{x}^{(1)} = 0\, \underline{x}^{(1)} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\lambda_2 = -1$ $\ominus$

2. limit as $t \to \infty$

   $e^{-t} \to 0$

   $\underline{x} \to c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

3. limit as $t \to -\infty$

   $\underline{x} \to$ parallel to $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$\boxed{\text{parallel lines}}$

NOTE: $\lambda_2 < 0 \to$ stop on $\underline{v}^{(1)}$

$\lambda_2 > 0 \to$ emanate from $v^{(1)}$

as $t \to \infty$, diverge

II. Real repeated λ:

(a) If λ is complete

geometric mult $=$ algebraic mult $k = 2$

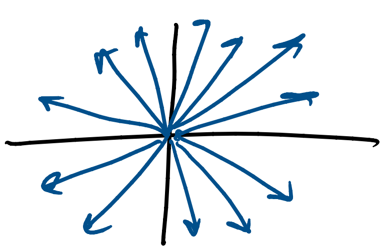
p

# of eigenvectors $= 2$

2 linearly independent eigenvectors $\underline{v}^{(1)}$ and $\underline{v}^{(2)}$

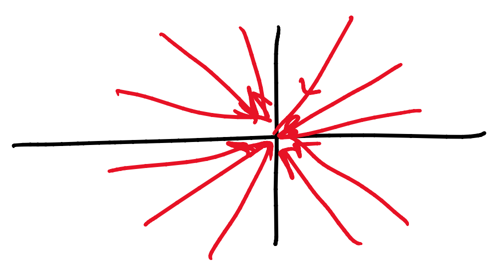
General Solution $\underline{x}(t) = c_1 e^{\lambda t} \underline{v}^{(1)} + c_2 e^{\lambda t} \underline{v}^{(2)}$

if $\lambda > 0$



proper nodal source

if $\lambda < 0$



proper nodal sink

<u>proper</u> <u>nodal</u> <u>source</u>

$$\underline{x}' = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \underline{x}$$

<u>proper</u> <u>nodal</u> <u>sink</u>

$$\underline{x}' = \begin{bmatrix} -3 & 0 \\ 0 & -3 \end{bmatrix} \underline{x}$$

(b) If λ is defective

geometric mult $<$ algebraic mult $k=2$

$p = 1$

$\#$ of eigenvectors $= 1$

Ex: $\underline{x}' = \begin{bmatrix} 4 & 1 \\ -1 & 2 \end{bmatrix} \underline{x}$ $\quad \lambda = 3$ with $k=2$

$\boxed{\lambda=3}$ $(\underline{A} - 3\underline{\underline{I}})\underline{v} = \underline{0}$

$$\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightarrow \quad v_1 + v_2 = 0, \quad v_2 = -v_1$$

one free variable $\longrightarrow$ one eigenvector $\rightarrow$ $p=1$

$$\underline{v} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$\lambda = 3$ is defective with defect $d = 1$

Generalized eigenvector $\underline{u}$

$$(\underline{A} - 3\underline{\underline{I}})\underline{u} = \underline{v}$$

$$\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \rightarrow \quad u_1 + u_2 = 1, \quad u_2 = 1 - u_1$$

choose $u_1 = 0$

$$\underline{u} = \begin{bmatrix} u_1 \\ 1 - u_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

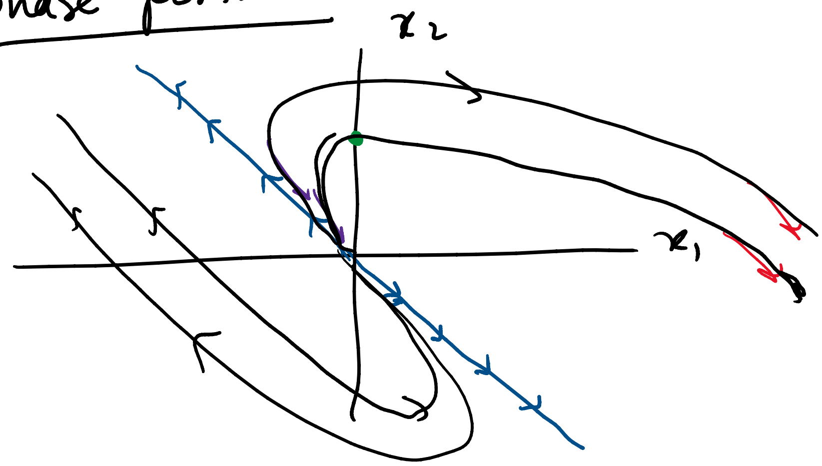
General solution:

$$\underline{x}(t) = c_1 e^{\lambda t} \underline{v} + c_2 e^{\lambda t} \{ t\underline{v} + \underline{u} \}$$

$$\underline{x}(t) = c_1 e^{\lambda t}\underline{v} + c_2 e^{\lambda t}[\ldots]$$

$$= c_1 e^{3t}\begin{bmatrix}1\\-1\end{bmatrix} + c_2 e^{3t}\left\{t\begin{bmatrix}1\\-1\end{bmatrix} + \begin{bmatrix}0\\1\end{bmatrix}\right\}$$

phase portrait:



1. Draw eigenvector $\underline{v}$
   $\lambda = 3$ (+)
2. let $c_1 = 0$, $c_2 = 1$
   $\hat{x} = te^{3t}\begin{bmatrix}1\\-1\end{bmatrix} + e^{3t}\begin{bmatrix}0\\1\end{bmatrix}$
   @ $t=0$ $\hat{x} = \begin{bmatrix}0\\1\end{bmatrix}$
3. as $t \to +\infty$
   te^{3t} dominates
   $\underline{x}$ parallel to $\begin{bmatrix}1\\-1\end{bmatrix}$
4. as $t \to -\infty$
   $e^{3t} \to 0$
   $te^{3t} \to 0$
   $\underline{x} \to \underline{0}$ parallel to $\begin{bmatrix}1\\-1\end{bmatrix}$

$$\lim_{t\to-\infty} te^{3t} = \lim_{t\to-\infty} \frac{t}{e^{-3t}}$$

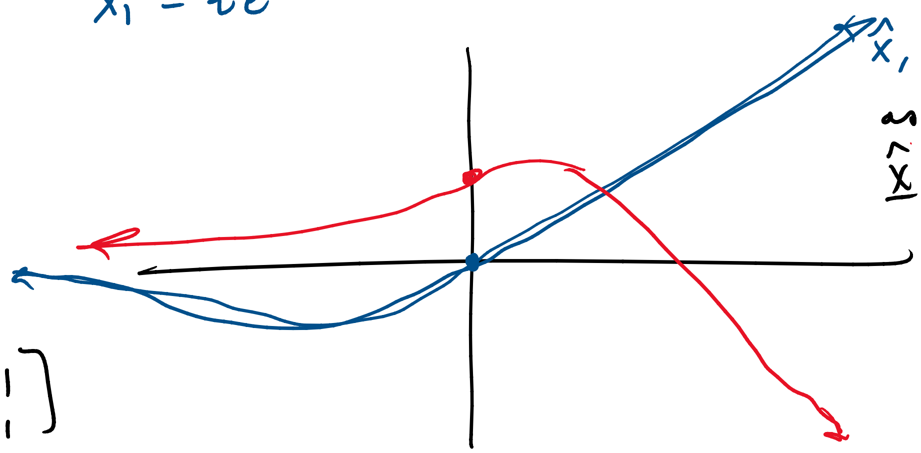
L'Hôpitals rule $= \lim_{t\to-\infty} \dfrac{1}{-3e^{-3t}} = 0^-$

improper nodal source

another way to look at this:

$\hat{\underline{x}} = \begin{bmatrix}\hat{x}_1(t)\\ \hat{x}_2(t)\end{bmatrix}$ $\quad \hat{x}_1 = te^{3t}$ $\quad \hat{x}_2 = -te^{3t} + e^{3t} = (1-t)e^{3t}$

as $t \to -\infty$
$\hat{\underline{x}} \to \begin{bmatrix}0\\0\end{bmatrix}$
along $\begin{bmatrix}-1\\1\end{bmatrix}$



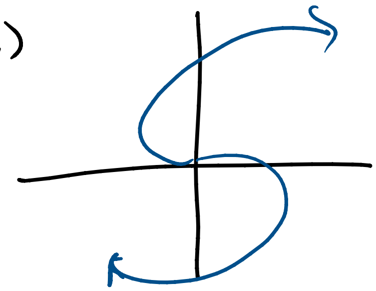
as $t \to \infty$
$\hat{\underline{x}} \to \begin{bmatrix}+\infty\\-\infty\end{bmatrix}$

III. Complex λ:

$$\lambda = a \pm bi$$

III. Complex λ: $\lambda = a \pm bi$

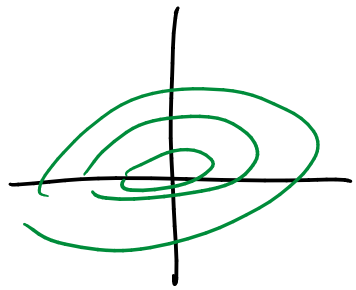
(a)



$a > 0$

spiral source

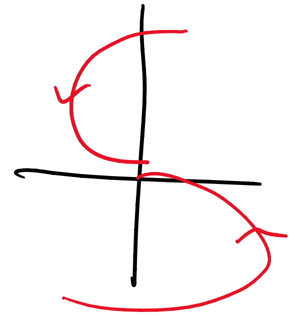
(b)



$a = 0$

center

(c)



$a < 0$

spiral sink

look at general solution to determine direction of rotation

$\lambda = a \pm ib$ $\quad \underline{x}(t) = e^{at}\{c_1 \underline{w} + c_2 \underline{v}\}$

evaluate $\begin{cases} \underline{w}(0) & \underline{w}'(0) \\ \underline{v}(0) & \underline{v}'(0) \end{cases}$

Ex: $\underline{x}' = \begin{bmatrix} 6 & -17 \\ 8 & -6 \end{bmatrix} \underline{x}$ $\quad \lambda = \pm 10i \longrightarrow$ center

$\lambda = 10i \quad \underline{v}^{(1)} = \begin{bmatrix} 3+5i \\ 4 \end{bmatrix}$

fund. soln:

$$\underline{x}^{(1)} = e^{10it}\begin{bmatrix} 3+5i \\ 4 \end{bmatrix} = \big(\cos(10t) + i\sin(10t)\big)\begin{bmatrix} 3+5i \\ 4 \end{bmatrix}$$

$$= \begin{bmatrix} 3\cos(10t) + 3i\sin(10t) + 5i\cos(10t) - 5\sin(10t) \\ 4\cos(10t) + 4i\sin(10t) \end{bmatrix}$$

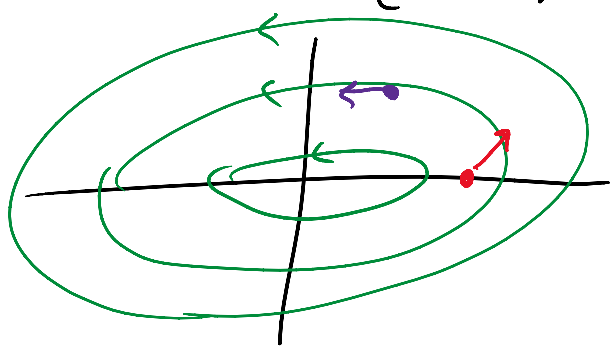
$$= \begin{bmatrix} 3\cos(10t) - 5\sin(10t) \\ 4\cos(10t) \end{bmatrix} + i\begin{bmatrix} 3\sin(10t) + 5\cos(10t) \\ 4\sin(10t) \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} \cdots \\ 4\cos(10t) \end{bmatrix}}_{\underline{w}} \cdots \underbrace{\begin{bmatrix} \cdots \\ 4\sin(10t) \end{bmatrix}}_{\underline{v}}$$

real valued solution:

$$\underline{x}(t) = C_1 \underline{w} + C_2 \underline{v}$$

$$= C_1 \begin{bmatrix} 3\cos(10t) - 5\sin(10t) \\ 4\cos(10t) \end{bmatrix} + C_2 \begin{bmatrix} 3\sin(10t) + 5\sin(10t) \\ 4\sin(10t) \end{bmatrix}$$



Q: arrows CW or CCW?

2. $\underline{w}(0) = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$ $\underline{w}'(0) = \begin{bmatrix} -50 \\ 0 \end{bmatrix}$

3. $\underline{v}(0) = \begin{bmatrix} 5 \\ 0 \end{bmatrix}$ $\underline{v}'(0) = \begin{bmatrix} 30 \\ 40 \end{bmatrix}$

arrows point in CCW direction,