

★ Sec 6.1

Stability in the Phase Plane

Warm up:

The linear system:

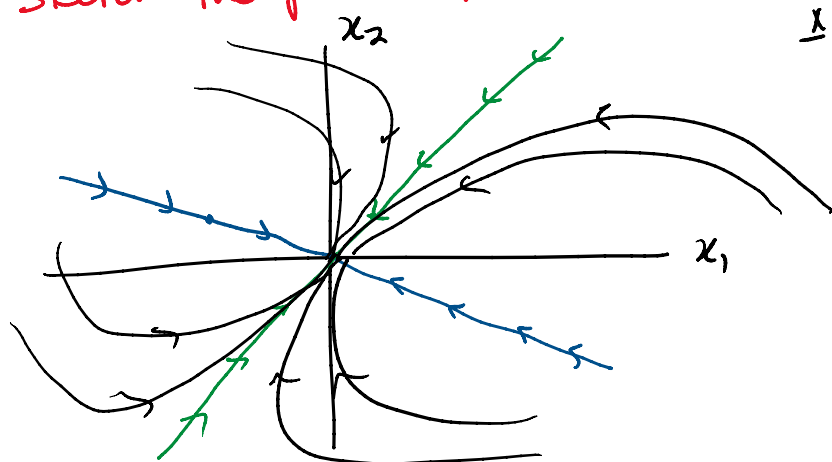
$$\underline{x}' = \begin{bmatrix} -4 & 3 \\ 1 & -2 \end{bmatrix} \underline{x}$$

has eigenvalues and eigenvectors

$$\left\{ \begin{array}{l} \lambda_1 = -5 \\ \underline{v}^{(1)} = \begin{bmatrix} -3 \\ 1 \end{bmatrix} \end{array} \right. \quad \left\{ \begin{array}{l} \lambda_2 = -1 \\ \underline{v}^{(2)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \end{array} \right.$$

Sketch the phase portrait and identify its type.

$$\underline{x} = c_1 e^{-5t} \begin{bmatrix} -3 \\ 1 \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$



improper
nodal
sink

I. Nonlinear Systems:

A system of equations

$$(*) \quad \begin{cases} x' = F(x, y) \\ y' = G(x, y) \end{cases}$$

is called autonomous if the RHS has no t -dependence

$$x' = \frac{dx}{dt}$$

Def: a critical point of system (*) is a point (x_*, y_*) where both

$$F(x_*, y_*) = 0 \quad \text{and} \quad G(x_*, y_*) = 0$$

point (x_*, y_*) when $\dots$
 $F(x_*, y_*) = 0$ and $G(x_*, y_*) = 0$

$$x' |_{(x_*, y_*)} = F(x_*, y_*) = 0$$

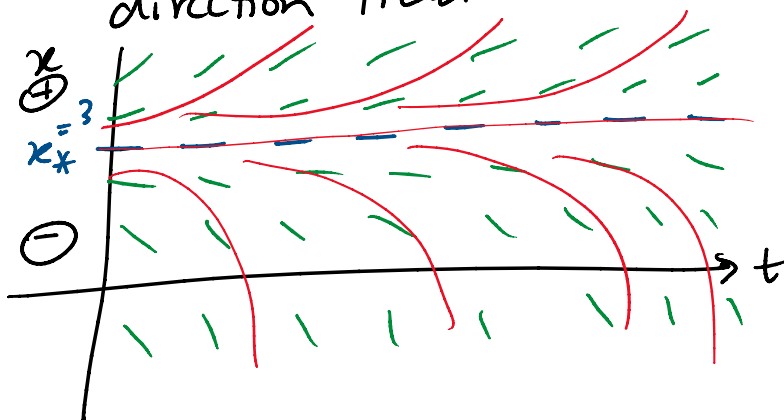
$$y' |_{(x_*, y_*)} = G(x_*, y_*) = 0$$

Recall: in 1D, we used critical points (c.p.) and direction fields to graphically represent solution curves

Ex: $x' = x - 3$

critical point
direction field

$$x_* - 3 = 0 \rightarrow x_* = 3$$



@ $x_* = 3$ $x' = 0$

if $x > 3$
 $x' = x - 3 > 0$ (+)

if $x < 3$
 $x' = x - 3 < 0$ (-)

Sketch solution curves

if $x(t) = x_* = 3$ $x' = \frac{d}{dt}(3) = 0$
 solves ODE $\rightarrow$ equilibrium solution

The solution curves diverge away from $x_* = 3$
 so the critical point $x_* = 3$ is unstable

GOAL: Do same thing in 2D

1D

2D

	1D	2D
critical point	x_*	(x_*, y_*)
graph	direction field x vs t	phase portraits y vs. x think of t as a parameterization ($y(t)$ vs. $x(t)$)

Ex:
$$\begin{cases} x' = x - 3 \\ y' = x + 5y + 2 \end{cases}$$

critical points:

$$x_* - 3 = 0$$

$$x_* = 3$$

$$x_* + 5y_* + 2 = 0$$

$$3 + 5y_* + 2 = 0$$

$$5y_* = -5$$

$$y_* = -1$$

So the critical point $(3, -1)$

Then, the constant-valued function:

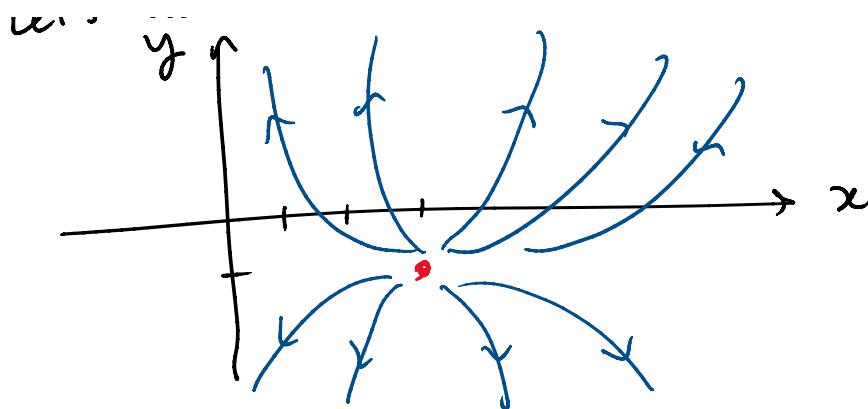
$$\begin{cases} x(t) = x_* = 3 \\ y(t) = y_* = -1 \end{cases}$$

Solve the system (*). We call this an equilibrium solution

Let's use a computer to calculate phase portrait:

$y \uparrow$ 1 2 3 4 5

critical point
at $(3, -1)$



critical point
at $(3, -1)$

@ c.p.
 $x' = 0$ and $y' = 0$

solution curves look like an improper nodal source
shifted from $(0,0) \rightarrow (3, -1)$

From this, since solution curves diverge away
from $(3, -1) \rightarrow$ we say it's unstable critical point

NOTE: We have seen already all the possible
behaviors of a c.p.

- (im) proper nodal source/sink
- saddle point
- spiral source/sink
- center
- parallel lines

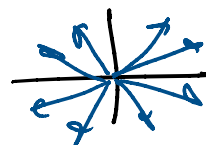
same in nonlinear systems

From the phase portrait, we can determine the
stability of the c.p.

Three types:

1. unstable: (some) solutions diverge away from
the critical point

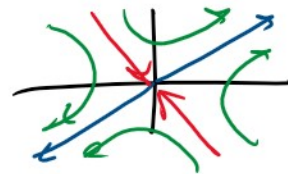
Ex: proper nodal
source



Ex: proper nodal source



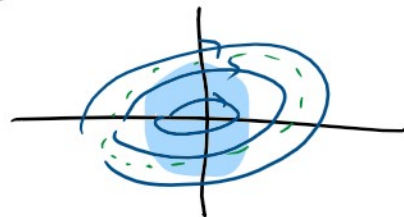
Ex: saddle point



2. Stable: solutions that start "close" to the critical point stay close to the c.p.

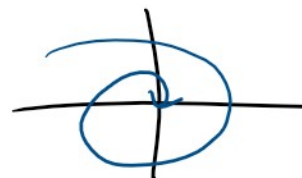
Ex: center

"don't diverge away"



3. Asymptotically stable: as $t \rightarrow \infty$, the solutions converge* to the critical point

Ex: spiral sink



*nuance: for asymptotically stable c.p.
NOT EVERY solution curve will converge to c.p.
BUT

if it starts "close enough" to (x_*, y_*)
the solution will converge to (x_*, y_*)

Ex:
$$\begin{cases} x' = 1 - y^3 \\ y' = x^2 - 4y \end{cases}$$

1. Find the critical points

$$\begin{aligned} 1 - y_*^3 &= 0 \\ 1 &= y_*^3 \\ \dots \end{aligned}$$

$$\begin{aligned} x_*^2 - 4y_* &= 0 \\ x_*^2 - 4 &= 0 \\ x_*^2 &= 4 \end{aligned}$$

$$1 = y^*$$

$$y^* = 1$$

$$x^* = -1$$

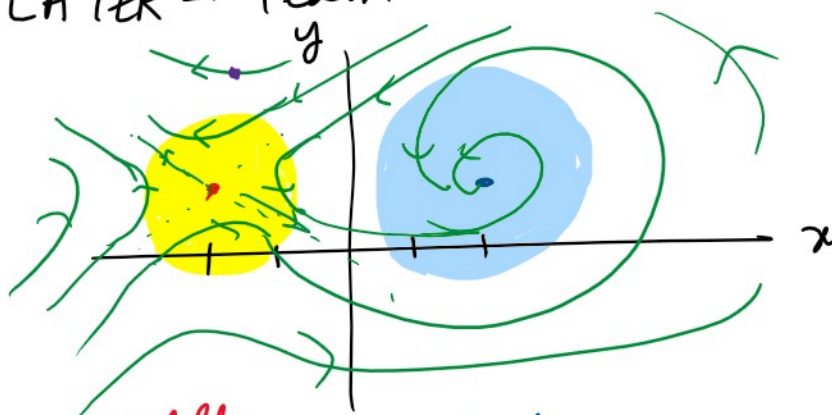
$$x_x^2 = 4$$

$$x_x = \pm 2$$

we have two critical points
 $(2, 1)$ and $(-2, 1)$

Nonlinear systems often have multiple c.p.

LATER $\rightarrow$ learn to draw phase portraits



given the
phase portrait

$(2, 1)$

$(-2, 1)$

Saddle
point
unstable

Spiral
sink
asymptotically
stable

Look for a
phase portrait
w/ c.p. at
 $(2, -1)$ and $(2, -1)$

We can use phase portraits and critical points
to analyze higher order scalar ODEs

Ex: $x'' + 2x + x^2 = 0$

scalar
2nd order
nonlinear

Convert to a 2nd order system;

let $x = x$, $y = x'$

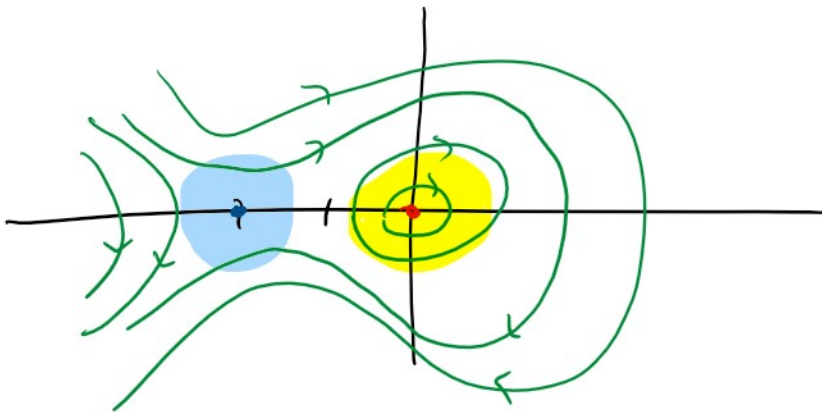
$$\begin{cases} x' = y \\ y' = x'' = -2x - x^2 \end{cases}$$

$$y' = x'' = -2x - x^2$$

1. Find the critical points
 $y_* = 0$

$$\begin{aligned} -2x_* - x_*^2 &= 0 \\ -x_*(2 + x_*) &= 0 \\ x_* &= 0, -2 \end{aligned}$$

critical points:
 $(0, 0)$ and $(-2, 0)$



$(0, 0)$ looks like a center
stable

$(-2, 0)$ looks like a saddle point
 $\rightarrow$ unstable

Exercise: let's sort all the possible behaviors
 by their stability:

- behaviors:
- (im)proper nodal source/sink
 - saddle point
 - spiral source/sink
 - center
 - parallel lines

unstable	stable	asymptotically stable
improper nodal source proper nodal source	center  parallel lines ($\lambda_2 < 0$)	(im)proper nodal sink spiral sink 

proper nodal source
 saddle point
 spiral source $\oplus$
 parallel lines ($\lambda_2 > 0$)

parallel lines ($\lambda_2 < 0$)

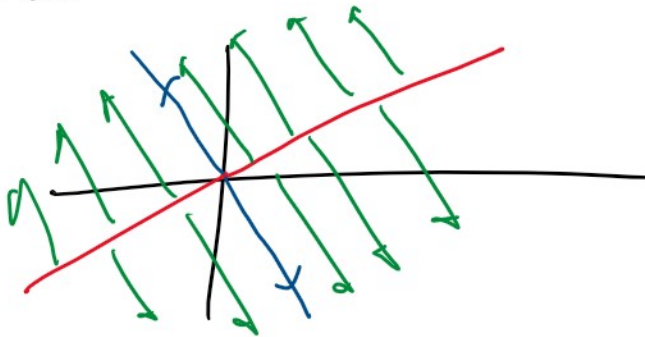
parallel lines ($\lambda_2 < 0$)

spiral sink $\ominus$

parallel lines stability:

2 cases: (i) $\lambda_1 = 0$ $\lambda_2 > 0$
 (ii) $\lambda_1 = 0$ $\lambda_2 < 0$

Case 1: $\lambda_2 > 0$ $\oplus$

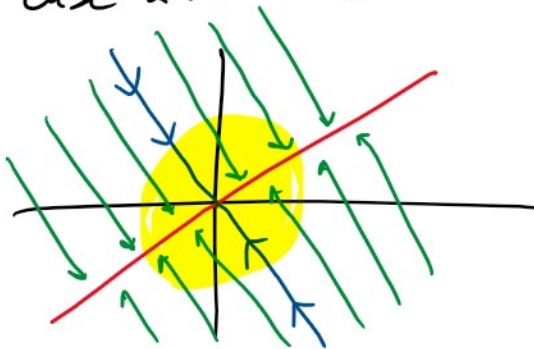


$$\lambda_1 = 0 \quad \underline{v}^{(1)}$$

$$\lambda_2 > 0 \quad \underline{v}^{(2)}$$

unstable

Case 2: $\lambda_2 < 0$ $\ominus$



$$\lambda_1 = 0 \quad \underline{v}^{(1)}$$

$$\lambda < 0 \quad \underline{v}^{(2)}$$

$$\ominus$$

stable