Lecture 8

☆ Section 6.2 - Part 2

Linear & Almost Linear Systems

Announcements:

Office Hours Today @ 2:30-3:30pm

Warm up: Find the Jacobian $\underline{\underline{J}}$ at $(0,0)$ of the nonlinear system:

$$x' = 4y + 7y^2 - 3x^5 y = F(x,y)$$
$$y' = x - 17y^5 + 6x^2 y = G(x,y)$$

$$\underline{\underline{J}} = \begin{bmatrix} F_x & F_y \\ G_x & G_y \end{bmatrix}\Bigg|_{(0,0)}$$

$$= \begin{bmatrix} -15x^4 y & 4 + 14y - 3x^5 \\ 1 + 12xy & -5(17)y^4 + 6x^2 \end{bmatrix}\Bigg|_{(0,0)}$$

$$\underline{\underline{J}} = \begin{bmatrix} 0 & 4 \\ 1 & 0 \end{bmatrix}$$

## I. Evaluating Almost Linear Systems:

An autonomous system:

$$x' = F(x,y)$$
$$y' = G(x,y)$$

has a critical point $(x_*, y_*)$ when

$$F(x_*, y_*) = 0 = G(x_*, y_*)$$

Use Taylor series expansion to get the almost linear system:

$$(*) \quad \begin{bmatrix} x' \\ y' \end{bmatrix} = \underbrace{\begin{bmatrix} F_x(x_*,y_*) & F_y(x_*,y_*) \\ G_x(x_*,y_*) & G_y(x_*,y_*) \end{bmatrix}}_{\underline{\underline{J}}\text{ - Jacobian matrix}} \begin{bmatrix} x - x_* \\ y - y_* \end{bmatrix} + \begin{bmatrix} r(x,y) \\ s(x,y) \end{bmatrix}$$

The associated linear system is:

$$(\square) \quad \underline{x}' = \underline{\underline{J}}\,\underline{x} \quad \text{where } \underline{x} = \begin{bmatrix} x - x_* \\ y - y_* \end{bmatrix}$$

Thm: The almost linear system $(*)$ will have ... type of critical point and the

Thm: The almost linear system (*) will have the same type of critical point and the same stability as the linear system (0)

UNLESS: (a) $\lambda_1 = \lambda_2$

type: node or spiral point
stable if $\lambda_1 = \lambda_2 < 0$
unstable if $\lambda_1 = \lambda_2 > 0$

(b) $\lambda = \pm bi$

type: center or spiral
may be either unstable, stable, or asymptotically stable

Ex: $\boxed{\begin{aligned} x' &= 4x + 7y \\ y' &= x - 2y \end{aligned}} \begin{aligned} &- x^2 + 3y^2 \quad r(x,y) \\ &- 13xy \quad s(x,y) \end{aligned}$

Taylor series

$$F(x,y) = \cancel{F(0,0)}^{\to 0} + F_x(0,0)x + F_y(0,0)y + r(x,y)$$

1. Claim: $(0,0)$ is a critical point

2. Calculate the Jacobian

$$\underline{\underline{J}} = \begin{bmatrix} F_x & F_y \\ G_x & G_y \end{bmatrix}\Bigg|_{(0,0)}$$

$$= \begin{bmatrix} 4-2x & 7+6y \\ 1-13y & -2-13x \end{bmatrix}\Bigg|_{(0,0)} = \begin{bmatrix} 4 & 7 \\ 1 & -2 \end{bmatrix}$$

Associated Linear system: $\begin{bmatrix} x' \\ y' \end{bmatrix} = \underline{\underline{J}}\,\underline{x} = \begin{bmatrix} 4 & 7 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$

keep all the linear terms from autonomous system

$$\boxed{\begin{aligned} x' &= 4x - 7y \\ y' &= x - 2y \end{aligned}}$$

3. Evaluate the phase portrait of the linear system:

$\begin{bmatrix} 4 & 7 \\ 1 & -2 \end{bmatrix}$ Char. eqn $\lambda^2 - T\lambda + D = 0$

$\lambda^2 - 2\lambda - 15 = 0$

$T = \text{trace}(\underline{\underline{J}}) = 4 - 2 = 2$

$D = \det(\underline{\underline{J}}) = -8 - 7 = -15$

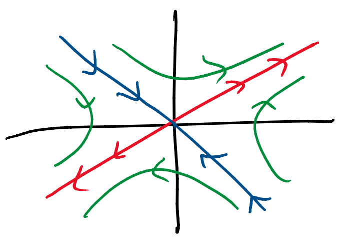
$(\lambda - 5)(\lambda + 3) = 0$

$\lambda = -3, +5$

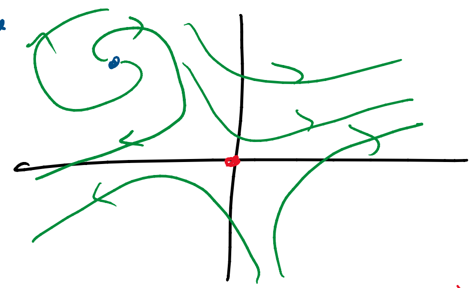
So $(0,0)$ is a saddle point and unstable for the linear system.

By the Thm, $(0,0)$ is also a saddle and unstable for the almost linear system.

Linear System



Almost Linear System




Ex: Find the critical point $(x_*, y_*)$ and apply the Thm to classify its type and stability.

$$x' = x - 2y$$
$$y' = 3x - 4y - 2$$

1. Find the c.p.

$$x - 2y = 0 \qquad\qquad 3x - 4y - 2 = 0$$
$$x = 2y \qquad\qquad 3(2y) - 4y = 2$$
$$2y = 2$$
$$x = 2 \qquad\qquad y = 1$$

$$(x_*, y_*) = (2,1)$$

2. Linearize $\longrightarrow$ Find Jacobian

$$\underline{\underline{J}} = \begin{bmatrix} F_x & F_y \\ G_x & G_y \end{bmatrix}\Bigg|_{(2,1)} = \begin{bmatrix} 1 & -2 \\ 3 & -4 \end{bmatrix}\Bigg|_{(2,1)} = \begin{bmatrix} 1 & -2 \\ 3 & -4 \end{bmatrix}$$

Associated Linear System

$$\underline{x}' = \begin{bmatrix} 1 & -2 \\ 3 & -4 \end{bmatrix} \underline{x}$$

3. Find the eigenvalues of $\underline{\underline{J}}$

$$\lambda^2 - T\lambda + D = 0$$
$$\lambda^2 + 3\lambda + 2 = 0$$
$$(\lambda + 2)(\lambda + 1) = 0$$

$T = tr(\underline{\underline{J}}) = 1 - 4 = -3$

$D = det(\underline{\underline{J}}) = -4 + 6 = 2$

$$(\lambda+2)(\lambda+1)=0$$
$$\lambda=-1,-2$$

4. Classify the c.p.
e.v. are real, neg $\longrightarrow$ improper nodal sink
and asymptotically stable

5. By the Thm, c.p. $(2,1)$ is a improper nodal sink and asymptoticaly stable

Thm: If $\lambda_1$ and $\lambda_2$ are eigenvalues of $\underline{x}'=\underline{A}\underline{x}$
1. $Re(\lambda)<0$ then asymptoticaly stable
2. $Re(\lambda)=0$ then stable
3. Otherwise unstable

Ex: The following system has a c.p. at $(0,0)$. Classify its type and stability using Thm 2.
$$x'=-6x+4y-x^2+3y^2$$
$$y'=-4x+2y-20xy$$

1. Linearize $\longrightarrow$ Find the Jacobian
$$\underline{\underline{J}}=\begin{pmatrix} F_x & F_y \\ G_x & G_y \end{pmatrix}\Bigg|_{(0,0)}=\begin{bmatrix} -6-2x & 4+6y \\ -4-20y & 2-20x \end{bmatrix}\Bigg|_{(0,0)}$$
$$=\begin{bmatrix} -6 & 4 \\ -4 & 2 \end{bmatrix}$$

Associated Linear System:
$$\underline{x}'=\begin{bmatrix} -6 & 4 \\ -4 & 2 \end{bmatrix}\underline{x}$$

2. Evaluate linear system $\longrightarrow$ calculate $\lambda$
$$\lambda^2-T\lambda+D=0$$
$T=tr(\underline{\underline{J}})=-6+2=-4$
$D=det(\underline{\underline{J}})=-12+16=4$
$$\lambda^2+4\lambda+4=0$$

$$\lambda^2 + 4\lambda + 4 = 0$$

$$(\lambda + 2)^2 = 0 \qquad \text{multiplicity}$$

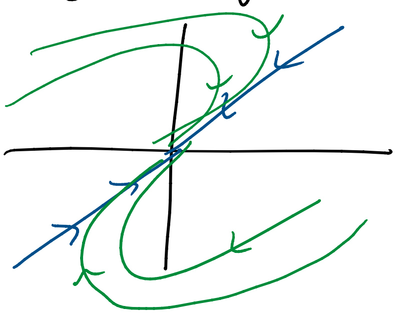
$$\lambda = -2 \quad \text{with} \quad k = 2$$

check that $\lambda$ is defective ✓

3. The linear system is an <u>improper</u> <u>nodal</u> <u>sink</u>
   b/c $\lambda = -2 < 0$ $\longrightarrow$ <u>asymptotically</u> <u>stable</u>

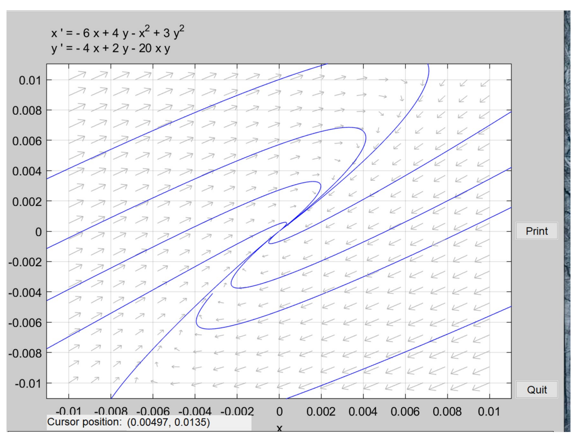
4. By Thm 2, because $\lambda_1 = \lambda_2$, all we can say is that $(0,0)$ is either a node or a spiral, but it still is <u>asymptotically</u> <u>stable</u>

Linear System



Almost Linear System

pplane 8 in MATLAB




zoom in on (0,0)

improper nodal sink

asymptotically stable

<u>Ex</u>: Find and classify all the critical points

$$\begin{cases} x' = y^2 - 1 \\ y' = x^3 - y \end{cases}$$

1. Find the critical points

$$y^2 - 1 = 0 \qquad\qquad x^3 - y = 0$$

$$y^2 = 1 \qquad\qquad x^3 = y$$

$$y = \pm 1 \qquad x^3 = +1 \qquad x^3 = -1$$

$$x = +1 \qquad x = -1$$

Two critical points: $(1,1)$ and $(-1,-1)$

2. Linearize around each c.p.

$$\underline{\underline{J}} = \begin{bmatrix} F_x & F_y \\ G_x & G_y \end{bmatrix} = \begin{bmatrix} 0 & 2y \\ 3x^2 & -1 \end{bmatrix}$$

@ (1,1) $\quad \underline{\underline{J}} = \begin{bmatrix} 0 & 2y \\ 3x^2 & -1 \end{bmatrix}\Bigg|_{(1,1)} = \begin{bmatrix} 0 & 2 \\ 3 & -1 \end{bmatrix}$

associated linear system: $\underline{x}' = \begin{bmatrix} 0 & 2 \\ 3 & -1 \end{bmatrix}\underline{x}$

@ (-1,-1) $\quad \underline{\underline{J}} = \begin{bmatrix} 0 & 2y \\ 3x^2 & -1 \end{bmatrix}\Bigg|_{(-1,-1)} = \begin{bmatrix} 0 & -2 \\ 3 & -1 \end{bmatrix}$

associated linear system: $\underline{x}' = \begin{bmatrix} 0 & -2 \\ 3 & -1 \end{bmatrix}\underline{x}$

3. Classify the c.p.

@ (1,1) $\quad \underline{\underline{J}} = \begin{bmatrix} 0 & 2 \\ 3 & -1 \end{bmatrix}$

Find the $\lambda$

$T = -1 \qquad D = -6$

$$\lambda^2 - T\lambda + D = 0$$
$$\lambda^2 + \lambda - 6 = 0$$
$$(\lambda - 2)(\lambda + 3) = 0$$
$$\lambda = 2, -3$$

The c.p. (1,1) is a saddle point and unstable

@ (-1,-1) $\quad \underline{\underline{J}} = \begin{bmatrix} 0 & -2 \\ 3 & -1 \end{bmatrix}$

$T = -1 \qquad D = 6$

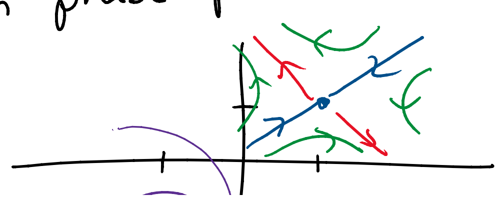
$$\lambda^2 + \lambda + 6 = 0$$
$$\lambda = \frac{-1}{2} \pm \frac{\sqrt{1^2 - 4\cdot 1\cdot 6}}{2}$$
$$= -\frac{1}{2} \pm \frac{1}{2}\sqrt{1-24}$$
$$= \frac{-1}{2} \pm \frac{i\sqrt{23}}{2}$$

The c.p. (-1,-1) is spiral sink

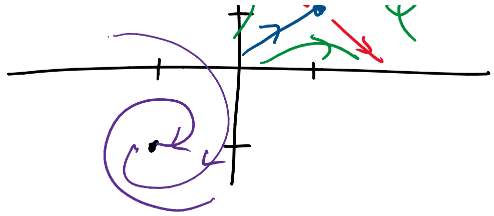
and since $Re(\lambda) = \frac{-1}{2} < 0$

asymptotically stable

sketch phase portrait



see more like this in next lecture



next lecture