Lecture 9

☆ Section 6.3

Ecological Models:
Predators & Competitors

Announcements:

- HW 5-8 + A2 due Tues Jun 29
- Midterm 1 on Thursday July 1
  - study guide
  - practice on BS
  - FAQ on Piazza
  - request review topics on Piazza

Warm up:

The autonomous system:

$$x' = 3x + 7y + 8xy = F(x,y)$$
$$y' = -x + 2y - 8xy = G(x,y)$$

has a critical point @ $(0,0)$
Find the associated linear system at $(0,0)$.

Find the Jacobian $\underline{\underline{J}} = \begin{bmatrix} F_x & F_y \\ G_x & G_y \end{bmatrix}$

$$\underline{\underline{J}}(0,0) = \begin{bmatrix} 3+8y & 7+8x \\ -1-8y & 2-8x \end{bmatrix}\Bigg|_{(0,0)} = \begin{bmatrix} 3 & 7 \\ -1 & 2 \end{bmatrix}$$

Associated linear system:

$$\underline{x}' = \begin{bmatrix} 3 & 7 \\ -1 & 2 \end{bmatrix} \underline{x}$$

I. Predator-Prey Model:

Consider a model of two species populations

$x(t)$ - population of prey (rabbits)
$y(t)$ - population of predators (foxes)

Alone, each population has exponential growth or decay

$\frac{dx}{dt} = ax$ — rabbits grow exp.

$\frac{dy}{dt} = -by$ — fox pop decay

(assume $a,b > 0$)

When both rabbits and foxes live in same habitat

- rabbit pop declines (eaten by foxes)
- fox pop. grows (increased food supply)

- rabbit pop declines (eaten by foxes)
- fox pop. grows (increased food supply)
- both changes are proportional to each other

Predator - prey Model:

$$\frac{dx}{dt} = ax - pxy = x(a - py)$$

$$\frac{dy}{dt} = \underbrace{-by}_{\text{exp growth/decay}} + \underbrace{qxy}_{\text{predation}} = y(-b + qx)$$

Ex: $\begin{cases} x' = 10x - 2xy \\ y' = -21y + 3xy \end{cases}$

Analyze this model $\longrightarrow$ Linearize around c.p.

1. Find the critical points

$10x - 2xy = 0$

$2x(5 - y) = 0$

$x = 0$ $\quad$ $y = 5$

$-21y + 3xy = 0$

$\boxed{x = 0 \quad -21y = 0 \quad y = 0}$

$-21(\cancel{5}) + 3x(\cancel{5}) = 0$

$3x = 21$

$\boxed{y = 5 \quad x = 7}$

Two critical points: $(0,0)$ and $(7,5)$

2. Find the Jacobian $\quad \underline{\underline{J}} = \begin{bmatrix} 10 - 2y & -2x \\ 3y & -21 + 3x \end{bmatrix}$

3. Linearize around each c.p. and classify its type and stability

@ $(0,0)$ $\quad J(0,0) = \begin{bmatrix} 10 & 0 \end{bmatrix}$

Linear system:

$\underline{x}' = \underline{\underline{J}}\,\underline{x}$

@ (0,0) $\qquad \underline{\underline{J}}(0,0) = \begin{bmatrix} 10 & 0 \\ 0 & -21 \end{bmatrix}$

Linear system:
$\underline{x}' = \underline{\underline{J}}\,\underline{x}$

Find the eigenvalues of $\underline{\underline{J}}$ $\qquad T = -11$, $D = -210$

$\lambda^2 - T\lambda + D = 0$

$\lambda^2 + 11\lambda - 210 = 0$

$(\lambda + 21)(\lambda - 10) = 0 \qquad \lambda = 10, -21$

So (0,0) is a saddle point
and because $\lambda = 10 > 0$ is $\oplus$ $\longrightarrow$ (0,0) is unstable

@ (7,5) $\qquad \underline{\underline{J}}(7,5) = \begin{bmatrix} 10 - 2\cdot 5 & -2\cdot 7 \\ 3\cdot 5 & -21 + 3\cdot 7 \end{bmatrix} = \begin{bmatrix} 0 & -14 \\ 15 & 0 \end{bmatrix}$

Find the eigenvalues $\quad \lambda = \pm i\sqrt{210}$

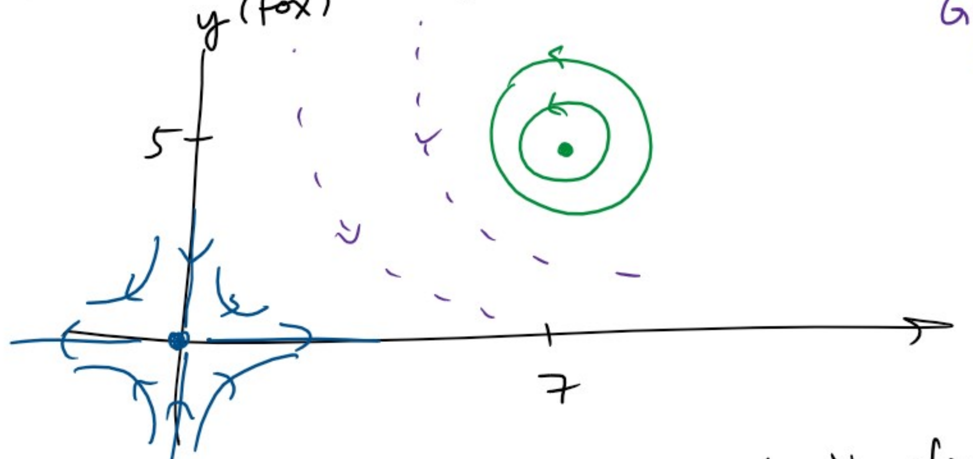
(7,5) is a center
and $Re(\lambda) = Re(\pm i\sqrt{210}) = 0 \longrightarrow$ stable

local approximation $\rightarrow$

(Need to check that this holds for nonlinear system)

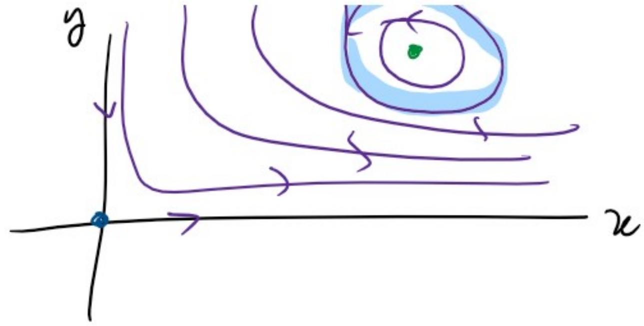
3. Sketch what the phase portrait looks like locally

Guess the solution curves in the middle



Check against the phase portrait for nonlinear system

compute w/ pplane 8

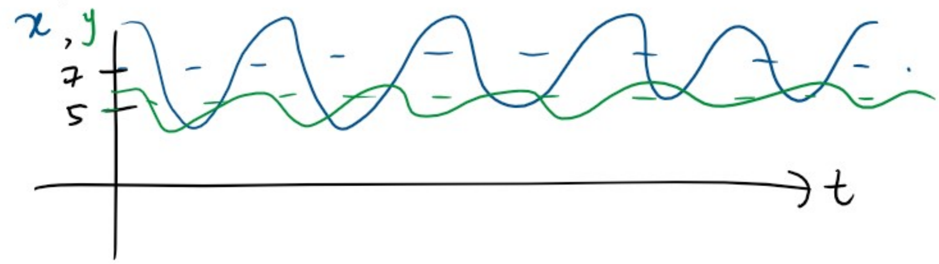


compute w/ pplane 8

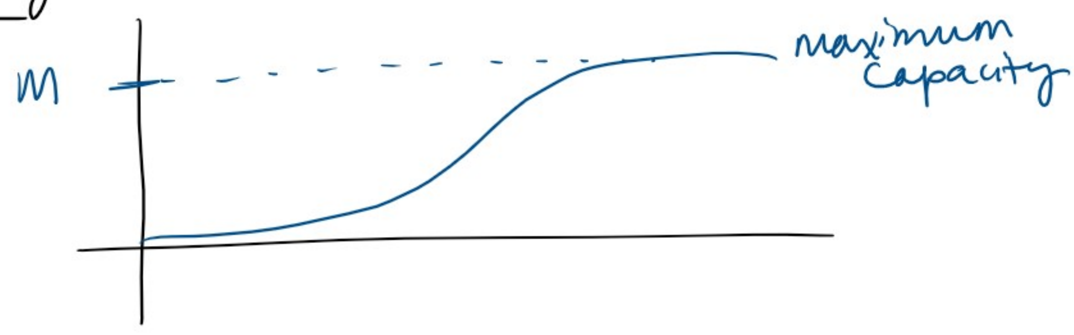
Q: What does this mean in terms of rabbits and foxes,

If at $t=0$ $x(0)>0$ and $y(0)>0$

then as $t\to\infty$, the populations will orbit around $(7,5)$



Now, let's suppose that rabbits $x(t)$ grow logistically $(x' = ax - bx^2)$



Q: How does logistic growth affect the predator-prey model?

Ex: $\begin{cases} x' = 10x - \frac{1}{2}x^2 - 2xy \\ y' = -21y + 3xy \end{cases}$

1. Find the critical points:

$$10x - \tfrac{1}{2}x^2 - 2xy = 0 \qquad\qquad -21y + 3xy = 0$$

$$3y(x-7) = 0$$

$$y = 0 \qquad\qquad x = 7$$

$$10x - \tfrac{1}{2}x^2 = 0$$

$$\tfrac{1}{2}x(20 - x) = 0$$

$$\boxed{x = 0 \qquad x = 20 \qquad y = 0}$$

$$10\cdot\cancel{7} - \tfrac{1}{2}(7)^{\cancel{2}} - 2\cdot\cancel{7}y = 0$$

$$\frac{13}{2} = \frac{20-7}{2} = 10 - \frac{7}{2} = 2y$$

$$\boxed{y = \tfrac{13}{4} \qquad x = 7}$$

Three critical points:

$$\boxed{\begin{array}{l}(0,0)\\(20,0)\\(7, \tfrac{13}{4})\end{array}}$$

2. Linearize around each c.p.
Find the Jacobian

$$\underline{\underline{J}} = \begin{bmatrix} 10 - x - 2y & -2x \\ 3y & -21 + 3x \end{bmatrix}$$

3. Evaluate the linear at each c.p.

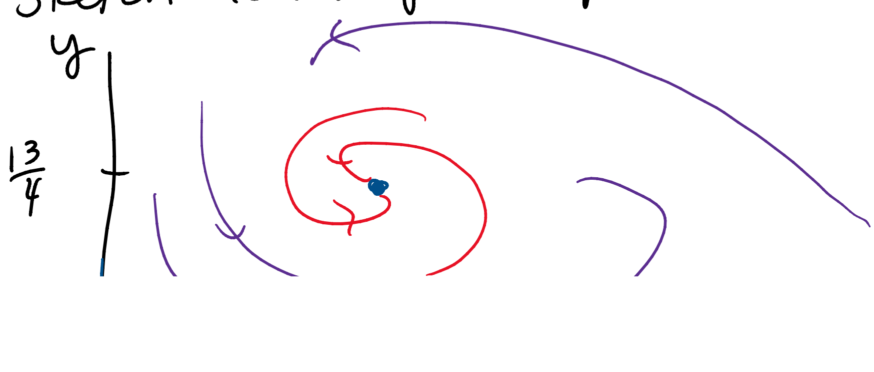
@ $(0,0)$ $\quad \underline{\underline{J}} = \begin{bmatrix} 10 & 0 \\ 0 & -21 \end{bmatrix}$ $\quad \lambda = -21, 10$ $\quad$ saddle point unstable

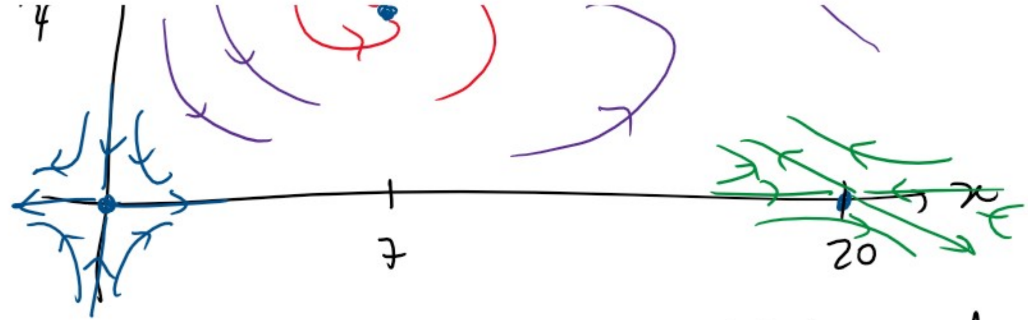
@ $(20,0)$ $\quad \underline{\underline{J}} = \begin{bmatrix} -10 & -40 \\ 0 & 39 \end{bmatrix}$ $\quad \lambda = 39, -10$ $\quad$ saddle point unstable

@ $(7, \tfrac{13}{4})$ $\quad \underline{\underline{J}} = \begin{bmatrix} -7/2 & -14 \\ 39/4 & 0 \end{bmatrix}$ $\quad \lambda = -\frac{7}{4} \pm i\frac{\sqrt{2135}}{4}$ $\quad$ spiral sink asymptotically stable

$\mathrm{Re}(\lambda) < 0$

4. Sketch local phase portraits





5. Interpret in terms of rabbits and foxes

as $t \to \infty$ $\quad x \longrightarrow 7$ (rabbits)

$y \longrightarrow \frac{13}{4}$ (foxes)

populations converge to a single value
"coexistence"

II. Competitors:

$x(t)$ - rabbits $\qquad y(t)$ - deer

- both eat vegetation
- neither preys on the other

} competition model

Equations:

$$\frac{dx}{dt} = \boxed{a_1 x - b_1 x^2} \boxed{- c_1 xy}$$

$$\frac{dy}{dt} = \boxed{a_2 y - b_2 y^2} \boxed{- c_2 xy}$$

logistic growth

competition for resources (both negative)

Note: $a_i, b_i, c_i > 0$

Rewrite:

$$x' = x(a_1 - b_1 x - c_1 y)$$
$$y' = y(a_2 - b_2 y - c_2 x)$$

This system has four critical points:

$\rightarrow (0,0)$

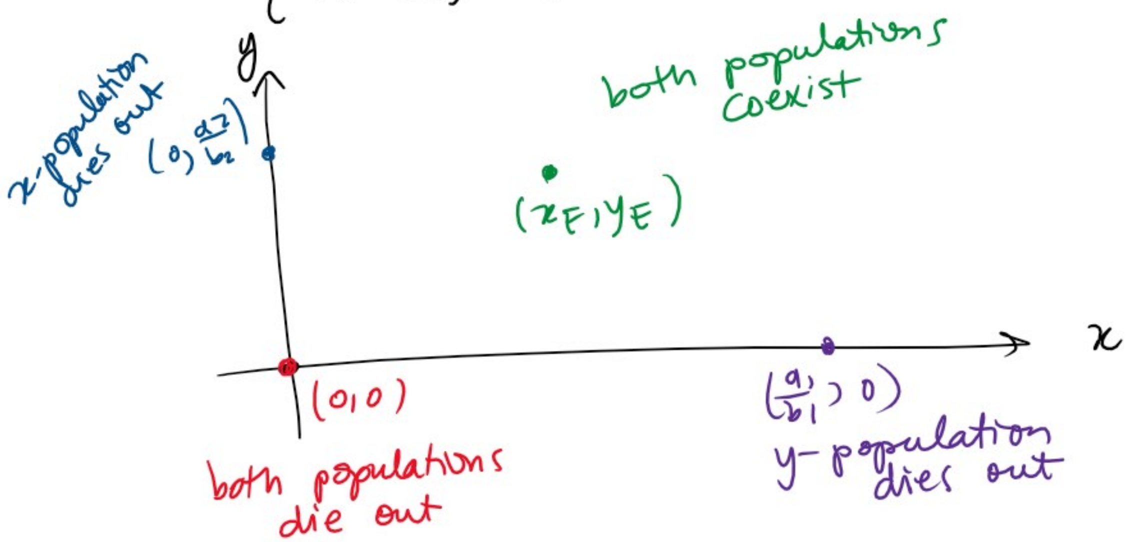
This system has __four__ critical p

$x=0$, and $y=0 \longrightarrow (0,0)$

if $x=0 \longrightarrow a_2 - b_2 y = 0 \longrightarrow \left(0, \frac{a_2}{b_2}\right)$

if $y=0 \longrightarrow a_1 - b_1 x = 0 \longrightarrow \left(\frac{a_1}{b_1}, 0\right)$

if $\begin{cases} a_1 - b_1 x - c_1 y = 0 \\ a_2 - b_2 y - c_2 x = 0 \end{cases}$ call the solution (x_E, y_E)



Often __coexistence__ is a goal.

__WANT__: (x_E, y_E) to be __stable__

__Coexistence Criteria:__

If $\underbrace{c_1 c_2}_{\text{competition}} < \underbrace{b_1 b_2}_{\text{inhibition}}$ then (x_E, y_E) is an asymptotically stable critical point

then the 2 species coexist.

← to calculate $\underline{I}$

__Ex:__

$$x' = 30x - 3x^2 + xy = x(30 - 3x + y)$$
$$y' = 60y - 3y^2 + 4xy = y(60 - 3y + 4x)$$

Find the critical point for coexistence: (x_E, y_E)

$$30 - 3x + y = 0$$

$$y = 3x - 30$$

$$60 - 3y + 4x = 0$$

$$60 - 3(3x - 30) + 4x = 0$$

$$60 - 9x + 90 + 4x = 0$$

$$150 = 5x$$

$$x = 30$$

$$y = 3(30) - 30$$

$$y = 60$$

$$\boxed{(x_E, y_E) = (30, 60)}$$

WANT: $(30, 60)$ to be stable or asymptotically stable

@ $(30, 60)$ $\underline{J} = \begin{bmatrix} -90 & 30 \\ 240 & -180 \end{bmatrix}$ $\lambda = -15(9 \pm \sqrt{41})$

$\lambda < 0$

improper nodal sink

asymptotically stable

$\rightarrow$ coexistence