Modeling Deformable Cells Using Spherical Harmonics

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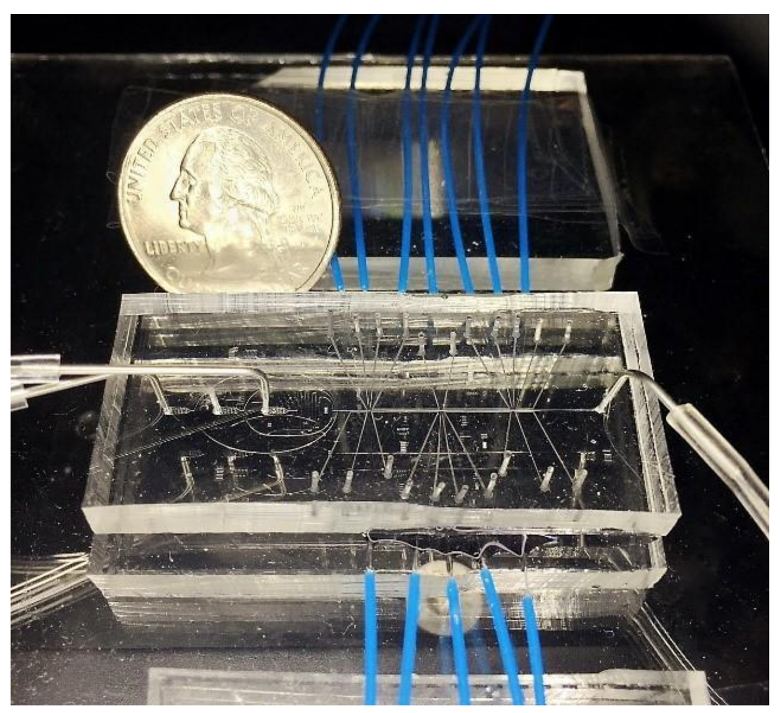
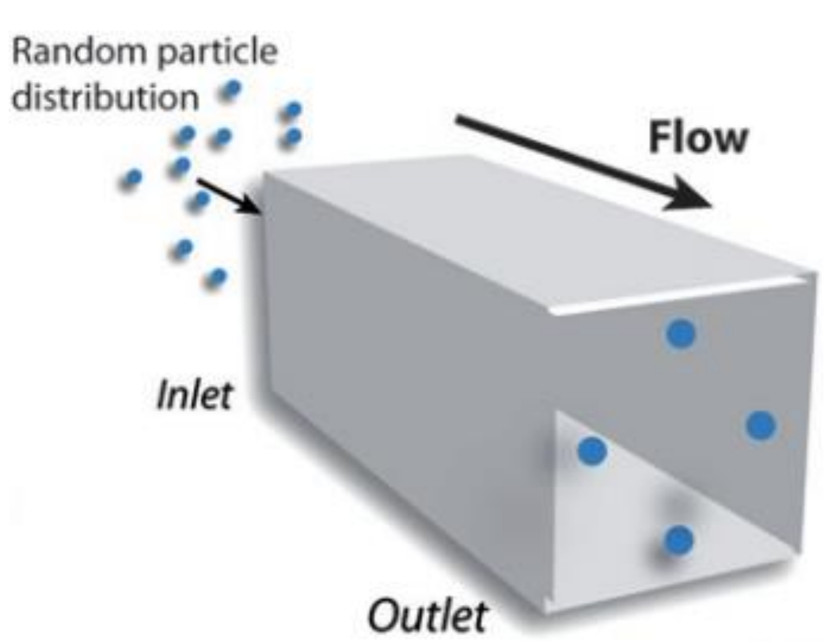
Background

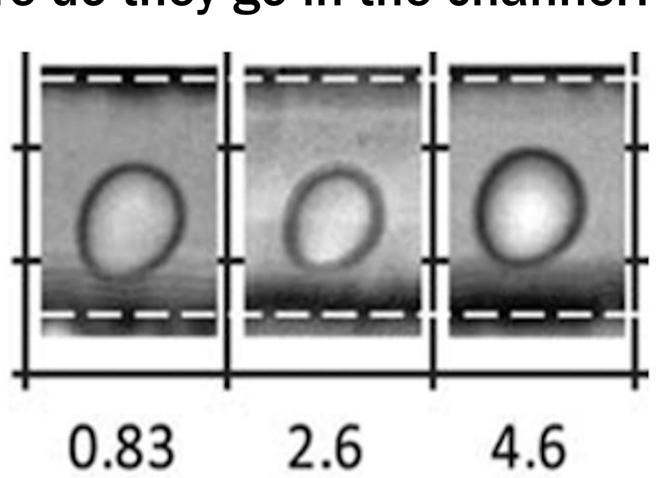
- Microfluidic devices allows for precise data collection of particles and cells
- Can be used for medical diagnostics. Examples include detecting cancer cells and counting rare cells in a blood test.



- A lot is known about the dynamics of plastic beads in a fluid chamber



- Human cells are deformable
- What shape do they take under flow and where do they go in the channel?

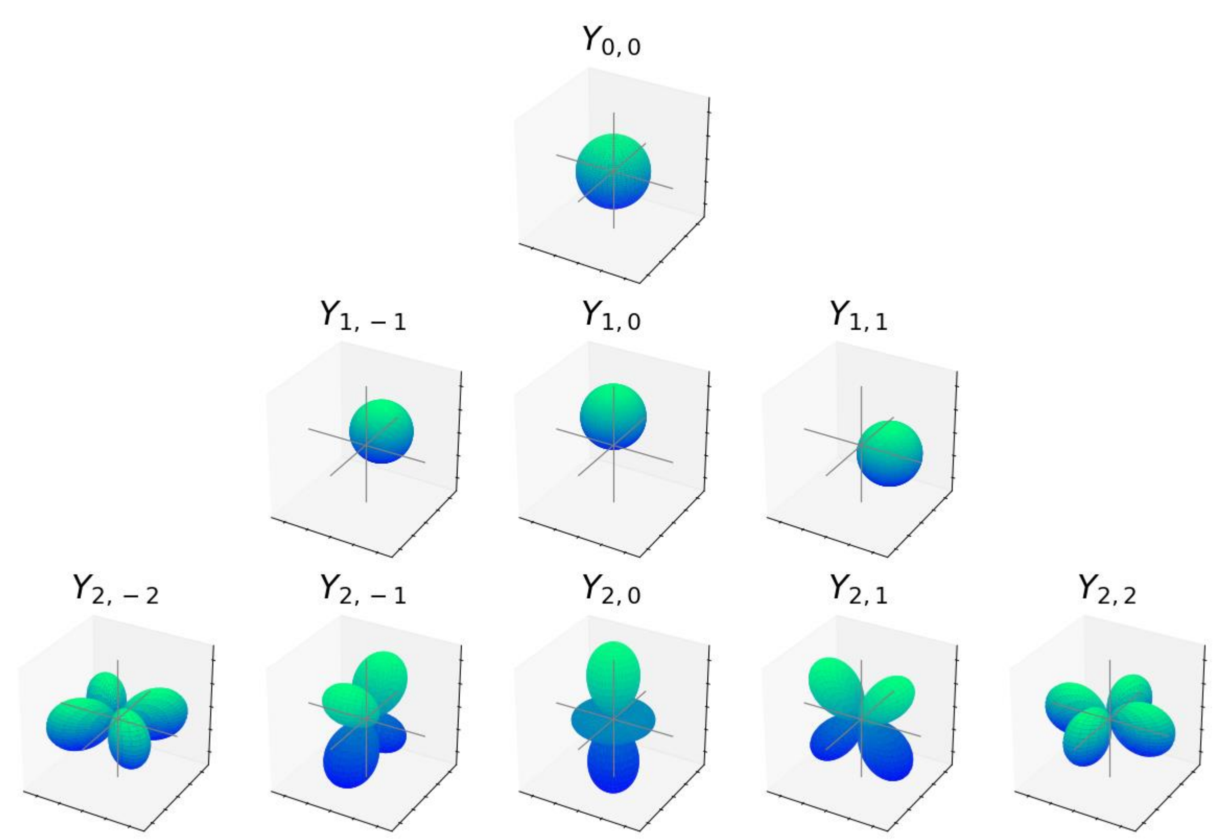


Viscosity Ratio λ

Q: What are the possible shapes?

- Represent shape as linear combination of spherical harmonics

$$r = f(\theta,\phi) = \sum_{\ell}\sum_{m} c_{\ell m} Y_\ell^m = \alpha Y_0^0 + \beta Y_1^0 + \gamma Y_1^1 + \delta Y_1^{-1}$$



- Apply constant volume constraint

Here $f(\theta,\phi)$ is equal to the radius of the surface

$$\mathrm{Vol}(f) = \int_0^{2\pi}\int_0^{\pi}\int_0^{f(\theta,\phi)} r^2\sin(\phi)\,dr\,d\phi\,d\theta = \frac{1}{3}\int_0^{2\pi}\int_0^{\pi}[f(\theta,\phi)]^3\sin(\phi)\,d\phi\,d\theta$$

Y_ℓ^m	Real-Valued Spherical Harmonic	Volume
Y_0^0	$\frac{1}{2\sqrt{\pi}}$	$\frac{1}{6\sqrt{\pi}}$
Y_1^{-1}	$\frac{1}{2}\sqrt{\frac{3}{\pi}}\sin(\phi)\sin(\theta)$	$\frac{1}{16}\sqrt{\frac{3}{\pi}}$
Y_1^0	$\frac{1}{2}\sqrt{\frac{3}{\pi}}\cos(\phi)$	$\frac{1}{16}\sqrt{\frac{3}{\pi}}$
Y_1^1	$\frac{1}{2}\sqrt{\frac{3}{\pi}}\sin(\phi)\cos(\theta)$	$\frac{1}{16}\sqrt{\frac{3}{\pi}}$
Y_2^{-2}	$\frac{1}{2}\sqrt{\frac{15}{\pi}}\sin^2(\phi)\cos(\theta)\sin(\theta)$	$\frac{4}{7}\sqrt{\frac{5}{3\pi^3}}$
Y_2^{-1}	$\frac{1}{2}\sqrt{\frac{15}{\pi}}\sin(\phi)\cos(\phi)\sin(\theta)$	$\frac{4}{7}\sqrt{\frac{5}{3\pi^3}}$
Y_2^0	$\frac{3}{4}\sqrt{\frac{5}{\pi}}\cos^2(\phi)$	$\frac{45}{112}\sqrt{\frac{5}{\pi}}$
Y_2^1	$\frac{1}{2}\sqrt{\frac{15}{\pi}}\sin(\phi)\cos(\phi)\cos(\theta)$	$\frac{4}{7}\sqrt{\frac{5}{3\pi^3}}$
Y_2^2	$\frac{1}{4}\sqrt{\frac{15}{\pi}}\sin^2(\phi)\cos(2\theta)$	$\frac{4}{7}\sqrt{\frac{5}{3\pi^3}}$

Constraint:

$$\frac{\alpha^3 + 3\alpha(\beta^2+\gamma^2+\delta^2)}{\sqrt{6\pi}} = V$$

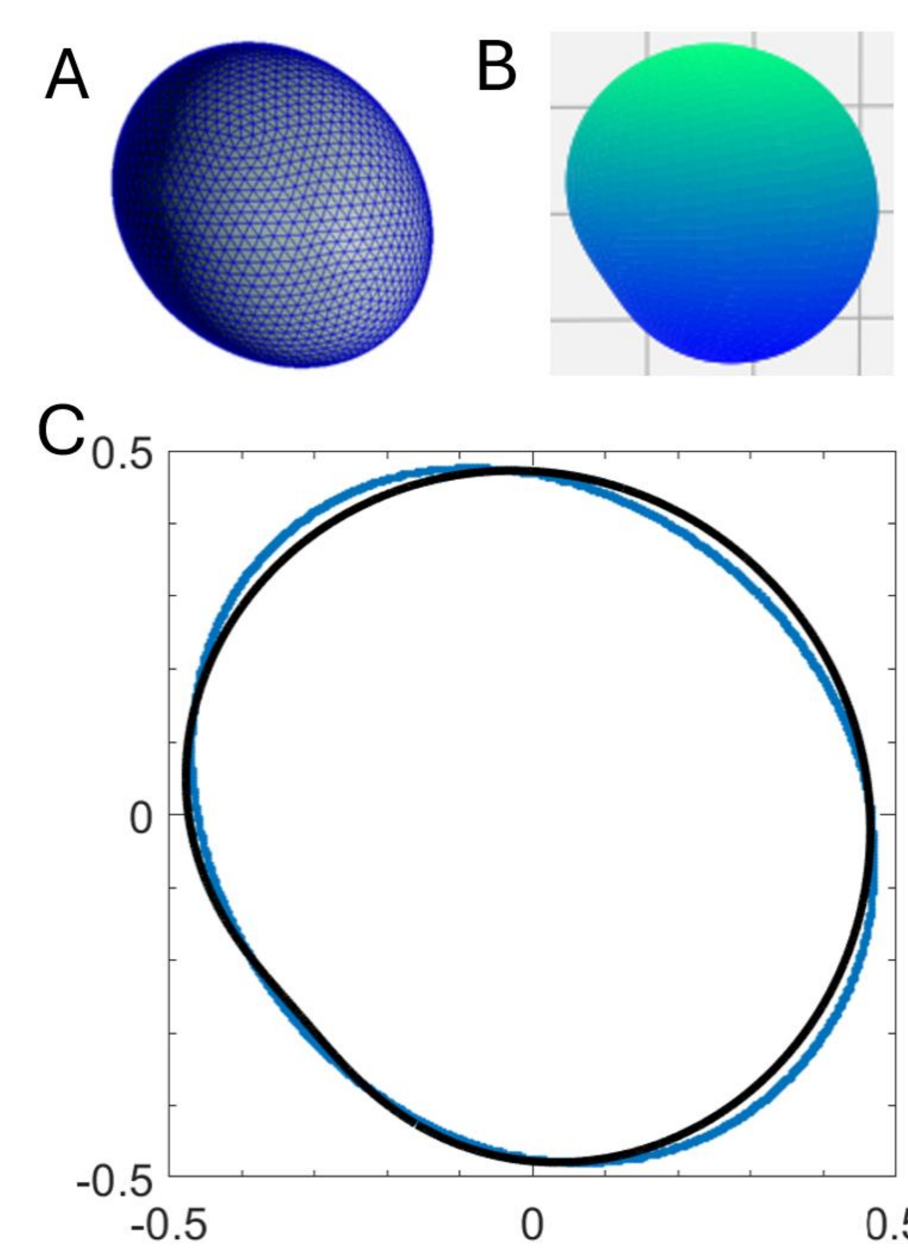
Given β, γ, δ, use Cardano's formula to find:

$$\alpha = \sqrt[3]{u_1} + \sqrt[3]{u_2}$$

Such that $u_1 = \frac{1}{2} + \sqrt{\frac{1}{4} + (\beta^2+\gamma^2+\delta^2)^3}$

and $u_2 = \frac{1}{2} + \sqrt{\frac{1}{4} - (\beta^2+\gamma^2+\delta^2)^3}$

Optimization



Algorithm Determining optimal $\alpha, \beta, \gamma, \delta$ for image input

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Perform edge detection and digitization on 2-D image
Compute center of mass and volume from digitized image
Initialize guesses for β, γ, δ
for each guess of β, γ, δ do
    Compute α
    Measure error between model and image
end for
Minimize error to determine optimal α, β, γ, δ
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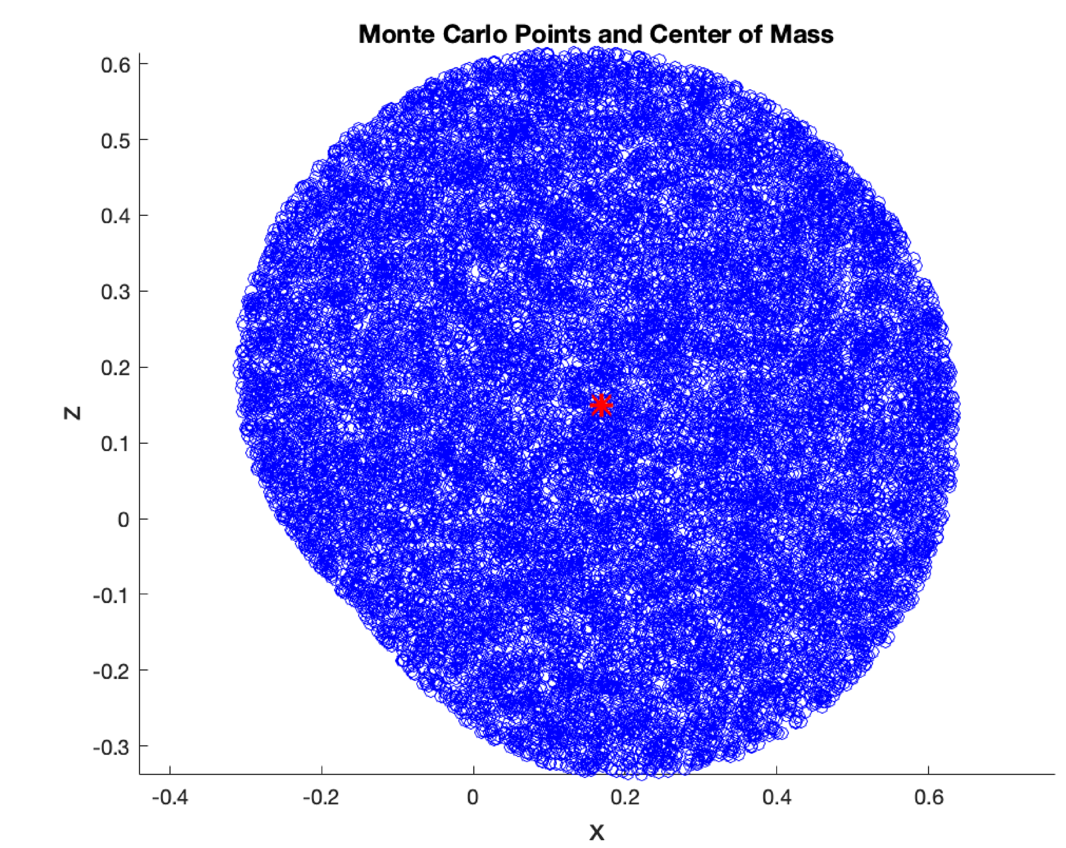
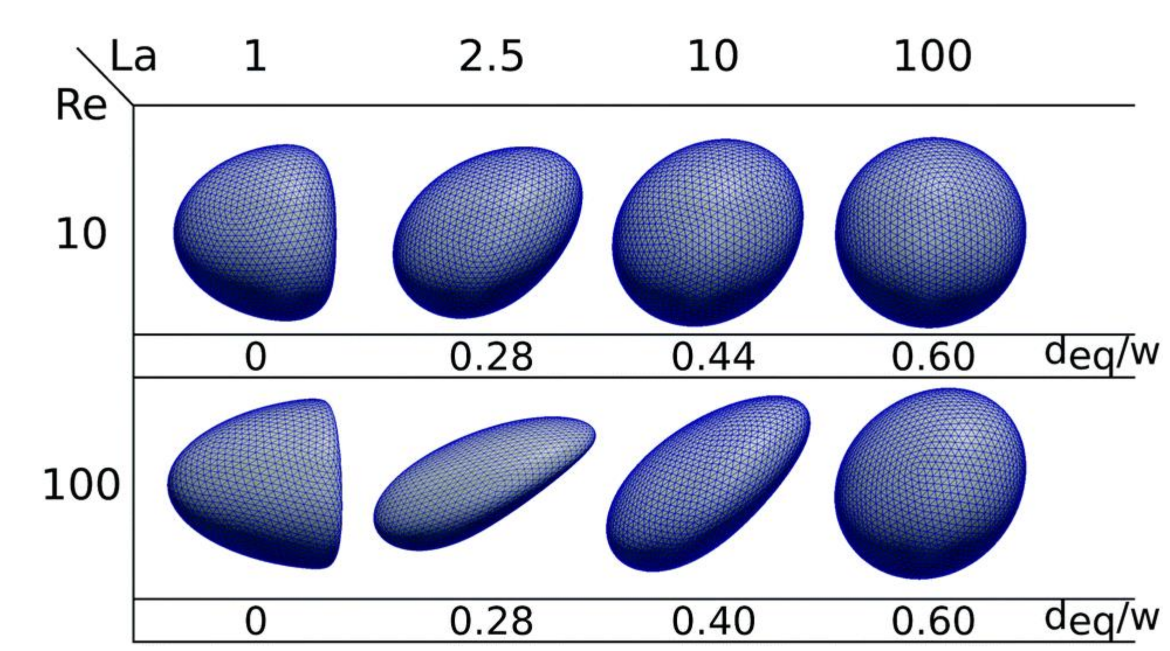
Monte Carlo Integration

- Normalizing the volume from 2-D data
- Used to find volume of 2-D sphere
- Finds the center of mass of spherical harmonic linear combinations in optimization



Future Work



- Model linear combinations with more terms
- Simulate flow around a shape with constant volume using the constraints
- Optimize coefficients to minimize drag

References

- *Gregory A Cooksey | NIST*. (2022b, December 8). NIST. https://www.nist.gov/people/gregory-cooksey
- Hur, S. C., Henderson-MacLennan, N. K., McCabe, E. R., & Di Carlo, D. (2011). Deformability-based cell classification and enrichment using inertial microfluidics. *Lab on a Chip*, *11*(5), 912-920.