Lesson 31: Intro to Matrices

Def. A matrix is an arrangement of numbers into rows and columns.

Ex.1 Given $A = \begin{bmatrix} 1 & 3 & 5 \\ 3 & -2 & 0 \\ -5 & -1 & 7 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 & 0 \\ -1 & 3 & 1 \\ 0 & 2 & -1 \end{bmatrix}$.

(a) Find $2A$. (Multiply each entry by 2)

$$2A = 2\begin{bmatrix} 1 & 3 & 5 \\ 3 & -2 & 0 \\ -5 & -1 & 7 \end{bmatrix} = \begin{bmatrix} 2(1) & 2(3) & 2(5) \\ 2(3) & 2(-2) & 2(0) \\ 2(-5) & 2(-1) & 2(7) \end{bmatrix} = \begin{bmatrix} 2 & 6 & 10 \\ 6 & -4 & 0 \\ -10 & -2 & 14 \end{bmatrix}$$

(b) Find $A + B$. (Add corresponding entries of A and B.)

$$A+B = \begin{bmatrix} 1 & 3 & 5 \\ 3 & -2 & 0 \\ -5 & -1 & 7 \end{bmatrix} + \begin{bmatrix} 1 & 2 & 0 \\ -1 & 3 & 1 \\ 0 & 2 & -1 \end{bmatrix} = \begin{bmatrix} 1+1 & 3+2 & 5+0 \\ 3+(-1) & -2+3 & 0+1 \\ -5+0 & -1+2 & 7+(-1) \end{bmatrix} = \begin{bmatrix} 2 & 5 & 5 \\ 2 & 1 & 1 \\ -5 & 1 & 6 \end{bmatrix}$$

(c) Find $2A - 3B$.

$$2A - 3B = \underbrace{\begin{bmatrix} 2 & 6 & 10 \\ 6 & -4 & 0 \\ -10 & -2 & 14 \end{bmatrix}}_{\text{from (a)}} - \begin{bmatrix} 3(1) & 3(2) & 3(0) \\ 3(-1) & 3(3) & 3(1) \\ 3(0) & 3(2) & 3(-1) \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 6 & 10 \\ 6 & -4 & 0 \\ -10 & -2 & 14 \end{bmatrix} - \begin{bmatrix} 3 & 6 & 0 \\ -3 & 9 & 3 \\ 0 & 6 & -3 \end{bmatrix} = \begin{bmatrix} 2-3 & 6-6 & 10-0 \\ 6-(-3) & -4-9 & 0-3 \\ -10-0 & -2-6 & 14-(-3) \end{bmatrix} = \begin{bmatrix} -1 & 0 & 10 \\ 9 & -13 & -3 \\ -10 & -8 & 17 \end{bmatrix}$$

* Multiplying Matrices

Ex2 $A = \begin{bmatrix} 1 & 0 & -2 \end{bmatrix}$ $B = \begin{bmatrix} -1 \\ 3 \\ 2 \end{bmatrix}$

Find (a) AB and (b) BA.

(a) $AB = \begin{bmatrix} 1 & 0 & -2 \end{bmatrix} \begin{bmatrix} -1 \\ 3 \\ 2 \end{bmatrix}$

A is a 1×3 matrix. B is a 3×1 matrix

need to match

# of rows in AB   # of columns in AB

Note: We say matrices are
# of rows - by- # of columns. ($\times$)

To get each element of AB, we multiply a row of A by a column of B.

$$AB = \begin{bmatrix} 1 & 0 & -2 \end{bmatrix} \begin{bmatrix} -1 \\ 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 1(-1) + 0(3) + (-2)(2) \end{bmatrix} = \begin{bmatrix} -1 + 0 - 4 \end{bmatrix} = \begin{bmatrix} -5 \end{bmatrix}$$

(b) $BA = \begin{bmatrix} -1 \\ 3 \\ 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & -2 \end{bmatrix}$ How many rows in BA? 3
How many columns in BA? 3

3×1 (rows in BA) 1×3 (columns in BA)   match

1st row of B   2nd column of A

$$BA = \begin{bmatrix} -1 \\ 3 \\ 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & -2 \end{bmatrix} = \begin{bmatrix} -1(1) & -1(0) & -1(-2) \\ 3(1) & 3(0) & 3(-2) \\ 2(1) & 2(0) & 2(-2) \end{bmatrix} = \begin{bmatrix} -1 & 0 & 2 \\ 3 & 0 & -6 \\ 2 & 0 & -4 \end{bmatrix}$$

3rd row of B   1st column of A

Notice that the entry in the 3rd row and 1st column of BA comes from multiplying the 3rd row of B with the 1st column of A.

Ex.3 $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ $B = \begin{bmatrix} -1 & 2 \\ 0 & -3 \end{bmatrix}$

(a) $AB = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} -1 & 2 \\ 0 & -3 \end{bmatrix} = \begin{bmatrix} 1(-1) + 2(0) & 1(2) + 2(-3) \\ 3(-1) + 4(0) & 3(2) + 4(-3) \end{bmatrix} = \begin{bmatrix} -1 & -4 \\ -3 & -6 \end{bmatrix}$

(b) $BA = \begin{bmatrix} -1 & 2 \\ 0 & -3 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} -1(1) + 2(3) & -1(2) + 2(4) \\ 0(1) + (-3)(3) & 0(2) + (-3)(4) \end{bmatrix} = \begin{bmatrix} 5 & 6 \\ -9 & -12 \end{bmatrix}$

Note: $AB \neq BA$

* Augmented Matrices

Ex.4 Write the augmented matrix for the system:

$$\begin{cases} 2x + 3y + z = 2 \\ 3y - 2z = 0 \\ 2x + y = -1 \end{cases}$$

$$\begin{array}{lcccccc} & \text{coefficient on } x & \text{coefficient on } y & \text{coefficient on } z & = & \text{constant} & \\ & & \downarrow & \downarrow & \downarrow & \downarrow & \\ \text{Equation 1} \rightarrow & \left[\quad 2\right. & 3 & 1 & \big| & 2 & \left.\right] \\ \text{Equation 2} \rightarrow & \quad 0 & 3 & -2 & \big| & 0 & \\ \text{Equation 3} \rightarrow & \left[\quad 2\right. & 1 & 0 & \big| & -1 & \left.\right] \end{array}$$

This is because $\begin{bmatrix} 2 & 3 & 1 \\ 0 & 3 & -2 \\ 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2x + 3y + z \\ 3y - 2z \\ 2x + y \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}$

* Row Operations

(1) Switch rows: $R_1 \leftrightarrow R_2$ means switch rows 1 and 2

(2) Multiply a row by a scalar:

$2R_1 \rightarrow R_1$ means multiply row 1 by 2.

(3) Add two rows:

$R_1 + R_2 \rightarrow R_2$ means row 1 is added to row 2.

Row 1 does <u>not</u> change. Row 2 <u>does</u> change.

Ex.5 For the matrix $\begin{bmatrix} 1 & 0 & 3 \\ -2 & 4 & 1 \\ 3 & 5 & 0 \end{bmatrix}$ perform $R_2 - 3R_1 \rightarrow R_2$

$R_2 - 3R_1 = [-2 \ 4 \ 1] - 3[1 \ 0 \ 3] = [-2 \ 4 \ 1] - [3 \ 0 \ 9] = [-5 \ 4 \ -8]$

Replace R_2 with $[-5 \ 4 \ -8]$.

$$\begin{bmatrix} 1 & 0 & 3 \\ -5 & 4 & -8 \\ 3 & 5 & 0 \end{bmatrix}$$

Ex.6 Consider the system $\begin{cases} 2x + y - z = 0 \\ 3x + 2y = 3 \\ -2x + y - 4z = -5 \end{cases}$

(a) Write the augmented matrix for the system.

$$\left[\begin{array}{ccc|c} 2 & 1 & -1 & 0 \\ 3 & 2 & 0 & 3 \\ -2 & 1 & -4 & -5 \end{array}\right]$$

(b) For the matrix in (a), perform the row operation $R_1 + R_3 \to R_3$.

$R_1 + R_3 = [2\ 1\ -1\ |\ 0] + [-2\ 1\ -4\ |\ -5] = [0\ 2\ -5\ |\ -5]$

Replace R_3 with $[0\ 2\ -5\ |\ -5]$

$$\left[\begin{array}{ccc|c} 2 & 1 & -1 & 0 \\ 3 & 2 & 0 & 3 \\ 0 & 2 & -5 & -5 \end{array}\right]$$

(c) Write the system of linear equations for the augmented matrix in (b).

$$\begin{bmatrix} 2 & 1 & -1 \\ 3 & 2 & 0 \\ 0 & 2 & -5 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2x + y - z \\ 3x + 2y \\ 2y - 5z \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ -5 \end{bmatrix}$$

$$\begin{cases} 2x + y - z = 0 \\ 3x + 2y = 3 \\ 2y - 5z = -5 \end{cases}$$

Ex.7 Let $M = \begin{bmatrix} 2 & 0 \\ -1 & 3 \end{bmatrix}$. Compute $M^2 - 2M$.

$$M^2 = \begin{bmatrix} 2 & 0 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 2(2)+0(-1) & 2(0)+0(3) \\ -1(2)+3(-1) & -1(0)+3(3) \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ -5 & 9 \end{bmatrix}$$

$$2M = 2\begin{bmatrix} 2 & 0 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 2(2) & 2(0) \\ 2(-1) & 2(3) \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ -2 & 6 \end{bmatrix}$$

$$M^2 - 2M = \begin{bmatrix} 4 & 0 \\ -5 & 9 \end{bmatrix} - \begin{bmatrix} 4 & 0 \\ -2 & 6 \end{bmatrix} = \begin{bmatrix} 4-4 & 0-0 \\ -5-(-2) & 9-6 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ -3 & 3 \end{bmatrix}$$

Ex.8 Given $A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 3 & 4 \\ 5 & 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 1 & 3 \\ 1 & 4 & 7 \\ 0 & 2 & 0 \end{bmatrix}$.

(a) $$AB = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 3 & 4 \\ 5 & 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & 1 & 3 \\ 1 & 4 & 7 \\ 0 & 2 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1(2)+0(1)+2(0) & 1(1)+0(4)+2(2) & 1(3)+0(7)+2(0) \\ 0(2)+3(1)+4(0) & 0(1)+3(4)+4(2) & 0(3)+3(7)+4(0) \\ 5(2)+0(1)+1(0) & 5(1)+0(4)+1(2) & 5(3)+0(7)+1(0) \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 5 & 3 \\ 3 & 20 & 21 \\ 10 & 7 & 15 \end{bmatrix}$$

(b) $$BA = \begin{bmatrix} 2 & 1 & 3 \\ 1 & 4 & 7 \\ 0 & 2 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 2 \\ 0 & 3 & 4 \\ 5 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2(1)+1(0)+3(5) & 2(0)+1(3)+3(0) & 2(2)+1(4)+3(1) \\ 1(1)+4(0)+7(5) & 1(0)+4(3)+7(0) & 1(2)+4(4)+7(1) \\ 0(1)+2(0)+0(5) & 0(0)+2(3)+0(0) & 0(2)+2(4)+0(1) \end{bmatrix}$$

$$= \begin{bmatrix} 17 & 3 & 11 \\ 36 & 12 & 25 \\ 0 & 6 & 8 \end{bmatrix}$$