Lesson 32: Solving Systems With Matrices

* Gaussian Elimination: Use row operations on augmented matrices to get it in the form

$$\begin{array}{cccc} x & y & z & \text{constants} \\ \end{array}$$
$$\left[\begin{array}{ccc|c} 1 & \# & \# & \# \\ 0 & 1 & \# & \# \\ 0 & 0 & 1 & \# \end{array}\right]$$

Ex. 1 Solve $\begin{cases} 2x + 6y = 10 \\ 3x + 5y = 11. \end{cases}$

$$\left[\begin{array}{cc|c} 2 & 6 & 10 \\ 3 & 5 & 11 \end{array}\right] \xrightarrow{\frac{1}{2}R_1 \to R_1} \left[\begin{array}{cc|c} 1 & 3 & 5 \\ 3 & 5 & 11 \end{array}\right]$$

$$\xrightarrow{R_2 - 3R_1 \to R_2} \left[\begin{array}{cc|c} 1 & 3 & 5 \\ 3-3(1) & 5-3(3) & 11-3(5) \end{array}\right] = \left[\begin{array}{cc|c} 1 & 3 & 5 \\ 0 & -4 & -4 \end{array}\right]$$

$$\xrightarrow{-\frac{1}{4}R_2 \to R_2} \left[\begin{array}{cc|c} 1 & 3 & 5 \\ 0 & 1 & 1 \end{array}\right]$$

Then $1 \cdot x + 3y = 5$ and $1 \cdot y = 1 \Rightarrow y = 1$

$x + 3(1) = 5 \Rightarrow x = 2$ $\quad \boxed{(2,1)}$

Ex. 2 Solve $\begin{cases} x + 3y - z = 7 \\ -2x - 4y + 6z = -14 \\ 3x + 7y - 6z = 20 \end{cases}$

$$\left[\begin{array}{ccc|c} 1 & 3 & -1 & 7 \\ -2 & -4 & 6 & -14 \\ 3 & 7 & -6 & 20 \end{array}\right] \xrightarrow{R_3 - 3R_1 \to R_3} \left[\begin{array}{ccc|c} 1 & 3 & -1 & 7 \\ -2 & -4 & 6 & -14 \\ 3-3(1) & 7-3(3) & -6-3(-1) & 20-3(7) \end{array}\right]$$

$$= \left[\begin{array}{ccc|c} 1 & 3 & -1 & 7 \\ -2 & -4 & 6 & -14 \\ 0 & -2 & -3 & -1 \end{array}\right]$$

$$\xrightarrow{R_2 + 2R_1 \to R_2} \left[\begin{array}{ccc|c} 1 & 3 & -1 & 7 \\ -2+2(1) & -4+2(3) & 6+2(-1) & -14+2(7) \\ 0 & -2 & -3 & -1 \end{array}\right]$$

$$= \left[\begin{array}{ccc|c} 1 & 3 & -1 & 7 \\ 0 & 2 & 4 & 0 \\ 0 & -2 & -3 & -1 \end{array}\right]$$

$$\xrightarrow{R_3+R_2\to R_3}\left[\begin{array}{ccc|c}1&3&-1&7\\0&2&4&0\\0&-2+2&-3+4&-1+0\end{array}\right]=\left[\begin{array}{ccc|c}1&3&-1&7\\0&2&4&0\\0&0&1&-1\end{array}\right]$$

$$\xrightarrow{\frac{1}{2}R_2\to R_2}\left[\begin{array}{ccc|c}1&3&-1&7\\0&1&2&0\\0&0&1&-1\end{array}\right]$$

$$\Rightarrow\begin{cases}x+3y-z=7\\ \quad y+2z=0\\ \quad\quad z=-1\end{cases}\Rightarrow\ y+2(-1)=0\ \Rightarrow y=2\qquad \Rightarrow x+3(2)-(-1)=7,\quad x=0\qquad \boxed{(0,2,-1)}$$

__Ex.3__ Solve
$$\begin{cases}\quad 2y+z=-8\\ x-2y-3z=0\\ -x+y+2z=3\end{cases}.$$

$$\left[\begin{array}{ccc|c}0&2&1&-8\\1&-2&-3&0\\-1&1&2&3\end{array}\right]\xrightarrow{R_1\leftrightarrow R_2}\left[\begin{array}{ccc|c}1&-2&-3&0\\0&2&1&-8\\-1&1&2&3\end{array}\right]$$

$$\xrightarrow{R_3+R_1\to R_3}\left[\begin{array}{ccc|c}1&-2&-3&0\\0&2&1&-8\\-1+1&1+(-2)&2+(-3)&3+0\end{array}\right]=\left[\begin{array}{ccc|c}1&-2&-3&0\\0&2&1&-8\\0&-1&-1&3\end{array}\right]$$

$$\xrightarrow{2R_3+R_2\to R_3}\left[\begin{array}{ccc|c}1&-2&-3&0\\0&2&1&-8\\0&2(-1)+2&2(-1)+1&2(3)+(-8)\end{array}\right]=\left[\begin{array}{ccc|c}1&-2&-3&0\\0&2&1&-8\\0&0&-1&-2\end{array}\right]$$

$$\xrightarrow{-R_3\to R_3}\left[\begin{array}{ccc|c}1&-2&-3&0\\0&2&1&-8\\0&0&1&2\end{array}\right]$$

$$\xrightarrow{\frac{1}{2}R_2\to R_2}\left[\begin{array}{ccc|c}1&-2&-3&0\\0&1&1/2&-4\\0&0&1&2\end{array}\right]$$

$$\Rightarrow\begin{cases}x-2y-3z=0\\ \quad y+\frac{1}{2}z=-4\\ \quad\quad z=2\end{cases}\Rightarrow\ y+\tfrac{1}{2}(2)=-4,\quad y=-5\qquad\Rightarrow x-2(-5)-3(2)=0,\quad x=-4\qquad\boxed{(-4,-5,2)}$$

__Ex.4__ Solve
$$\begin{cases}x+y+z=6\\ 2x-y+z=3\\ x\quad\ +z=4\end{cases}$$

On your own: 1. Make augmented matrix

2. Perform the sequence of row operations:

(a) $-2R_1+R_2 \to R_2$

(b) $-R_1+R_3 \to R_3$

(c) $-R_2 \to R_2$; $-R_3 \to R_3$

(d) $R_2 \leftrightarrow R_3$

(e) $-3R_2+R_3 \to R_3$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 2 & -1 & 1 & 3 \\ 1 & 0 & 1 & 4 \end{array}\right] \xrightarrow[(a)]{-2R_1+R_2\to R_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 2-2(1) & -1-2(1) & 1-2(1) & 3-2(6) \\ 1 & 0 & 1 & 4 \end{array}\right]$$

$$= \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & -3 & -1 & -9 \\ 1 & 0 & 1 & 4 \end{array}\right]$$

$$\xrightarrow[(b)]{-R_1+R_3\to R_3} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & -3 & -1 & -9 \\ 1-(1) & 0-(1) & 1-(1) & 4-(6) \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & -3 & -1 & -9 \\ 0 & -1 & 0 & -2 \end{array}\right]$$

$$\xrightarrow[(c)]{-R_2\to R_2;\ -R_3\to R_3} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 3 & 1 & 9 \\ 0 & 1 & 0 & 2 \end{array}\right]$$

$$\xrightarrow[(d)]{R_2\leftrightarrow R_3} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 0 & 2 \\ 0 & 3 & 1 & 9 \end{array}\right]$$

$$\xrightarrow[(e)]{-3R_2+R_3\to R_3} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 0 & 2 \\ 0 & 3-3(1) & 1-3(0) & 9-3(2) \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

$$\Rightarrow \begin{cases} x+y+z=6 \\ y=2 \\ z=3 \end{cases} \quad \Rightarrow x+2+3=6 \qquad x=1$$

$$\boxed{(1,2,3)}$$

Ex.5 Solve $\begin{cases} x+y+z=5 \\ 2x+3y+5z=8 \\ 4x +5z=2 \end{cases}$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 2 & 3 & 5 & 8 \\ 4 & 0 & 5 & 2 \end{array}\right] \xrightarrow{R_2-2R_1\to R_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 2-2(1) & 3-2(1) & 5-2(1) & 8-2(5) \\ 4 & 0 & 5 & 2 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 0 & 1 & 3 & -2 \\ 4 & 0 & 5 & 2 \end{array}\right]$$

$$\xrightarrow{R_3-4R_1\to R_3} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 0 & 1 & 3 & -2 \\ 4-4(1) & 0-4(1) & 5-4(1) & 2-4(5) \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 0 & 1 & 3 & -2 \\ 0 & -4 & 1 & -18 \end{array}\right]$$

$$\xrightarrow{R_3+4R_2\to R_3}\left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 0 & 1 & 3 & -2 \\ 0 & -4+4(1) & 1+4(3) & -18+4(-2) \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 0 & 1 & 3 & -2 \\ 0 & 0 & 13 & -26 \end{array}\right]$$

$$\xrightarrow{\frac{1}{13}R_3\to R_3}\left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 0 & 1 & 3 & -2 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

$$\Rightarrow \begin{cases} x + y + z = 5 \\ \quad y + 3z = -2 \\ \qquad z = -2 \end{cases} \qquad y + 3(-2) = -2 \qquad x + 4 + (-2) = 5$$

$$y = 4 \qquad x = 3$$

$$\boxed{(3, 4, -2)}$$