Lesson 36: Finding Limits Analytically

* **Steps**: 1. Simplify function as much as possible.
Rewrite as a single fraction,
factor, cancel/simplify, etc.

2. Plug in the x-value and evaluate.
There are 4 possible outcomes:

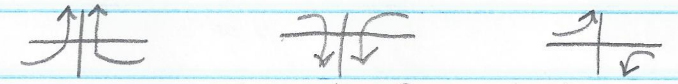
(a) limit = a number, then done.

"(b)" limit = $\frac{0}{\text{number}} = 0$, so done.

(c) limit = $\frac{\text{number}}{0}$.

As the denominator gets very small and close to 0, $\frac{\text{number}}{0}$ gets very big.
Then the options for this limit are:

∞ $-\infty$ DNE



Look at the left and right limits to decide if the limit is ∞, $-\infty$ or DNE.

(d) limit = $\frac{0}{0}$, then we need to manipulate the function some more. Rationalize square roots, factor, etc.

Ex. 1 Find $\lim_{x\to 0} (x^3 + 2x - 3)$.
Can't factor or simplify at all, so just plug in $x = 0$.
$\lim_{x\to 0} (x^3 + 2x - 3) = 0^3 + 2(0) - 3 = \boxed{-3}$

Ex. 2 Find $\lim_{x\to 2} \frac{x-2}{x^2+4}$.
Can't factor x^2+4 or simplify, so plug in $x = 2$.
$\lim_{x\to 2} \frac{x-2}{x^2+4} = \frac{2-2}{2^2+4} = \frac{0}{8} = \boxed{0}$

Ex. 3 Find $\lim_{x\to \frac{1}{2}} \frac{1}{2x-1}$.
Can't simplify, so $\lim_{x\to \frac{1}{2}} \frac{1}{2x-1} = \frac{1}{2(\frac{1}{2})-1} = \frac{1}{0}$.
Now we need to look at left and right limits.

$\lim_{x\to \frac{1}{2}^-} \frac{1}{2x-1}$

$x \to \frac{1}{2}^- \Rightarrow x < \frac{1}{2}$
$\Rightarrow 2x < 1$
$\Rightarrow 2x-1 < 0$.

Then the denom is negative and very small, so $\lim_{x\to \frac{1}{2}^-} \frac{1}{2x-1} = -\infty$
(Alternatively, pick a number on the left of and close to $\frac{1}{2}$: $x = 0.49$, $\frac{1}{2(0.49)-1} = -50$)

$\lim_{x\to \frac{1}{2}^+} \frac{1}{2x-1}$

$x \to \frac{1}{2}^+ \Rightarrow x > \frac{1}{2}$
$\Rightarrow 2x > 1$
$\Rightarrow 2x-1 > 0$

Then the denom is positive and very small, so $\lim_{x\to \frac{1}{2}^+} \frac{1}{2x-1} = \infty$.
(Or look at something like $x = 0.51$, $\frac{1}{2x-1} = 50$.)

Since the limits don't match, $\boxed{\text{DNE}}$

Ex. 4 Find $\lim_{x\to 1} \frac{1}{x^2-2x+1}$.

$\lim_{x\to 1} \frac{1}{x^2-2x+1} = \lim_{x\to 1} \frac{1}{(x-1)^2} = \frac{1}{0}$.

Need to look at left and right limits.

$\lim_{x\to 1^-} \frac{1}{(x-1)^2}$

$(x-1)^2$ is always positive, so $\lim_{x\to 1^-} \frac{1}{(x-1)^2} = \infty$

$\lim_{x\to 1^+} \frac{1}{(x-1)^2}$

$(x-1)^2$ is always positive, so $\lim_{x\to 1^+} \frac{1}{(x-1)^2} = \infty$

Then $\lim_{x\to 1} \frac{1}{(x-1)^2} = \infty$.

Note: Looking at examples 4 and 5, think back to even/odd multiplicity of roots. For $2x-1$, $x=\frac{1}{2}$ is a root with odd multiplicity, so on one side of the root, the sign will be different than on the other. Then one of the one-sided limits will go to ∞ and the other will go to $-\infty$, so the limit does not exist at the root.

For $(x-1)^2$, $x=1$ is a root with even multiplicity, so the sign of the function is the same on both sides of the root. Then the limit is either ∞ or $-\infty$ depending on the sign.

Ex. 5 Consider $f(x) = \frac{x^2+2x}{x^3-4x} = \frac{x(x+2)}{x(x+2)(x-2)} = \frac{1}{x-2}$.

(a) $\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{1}{x-2} = \frac{1}{0-2} = \boxed{-\frac{1}{2}}$

(If the unsimplified form, we get $\frac{0}{0}$.)

(b) $\lim_{x \to -2} f(x) = \lim_{x \to -2} \frac{1}{x-2}$ (In the unsimplified,
$= \frac{1}{-2-2} = \boxed{-\frac{1}{4}}$ we get $\frac{0}{0}$.)

(c) $\lim_{x \to 2} f(x) = \lim_{x \to 2} \frac{1}{x-2} = \frac{1}{2-2} = \frac{1}{0}$
$x = 2$ is a root of $x-2$ with odd multiplicity, so $\boxed{\text{DNE}}$
(Otherwise check left and right limits.)

Ex. 6 Consider $f(x) = \frac{x^3 - 3x^2}{x^3 + x^2} = \frac{x^2(x-3)}{x^2(x+1)} = \frac{x-3}{x+1}$.

(a) $\lim_{x \to 0} \frac{x-3}{x+1} = \frac{0-3}{0+1} = \boxed{-3}$

(b) $\lim_{x \to 3} \frac{x-3}{x+1} = \frac{3-3}{3+1} = \frac{0}{4} = \boxed{0}$

(c) $\lim_{x \to -1} \frac{x-3}{x+1} = \frac{-1-3}{-1+1} = \frac{-4}{0}$

We're only concerned with the sign of the denominator. $x = -1$ is a root of $x+1$ with odd multiplicity, so $\boxed{\text{DNE}}$.

Ex. 7 Consider $f(x) = \frac{x^2 - 4}{x^2 - 3x + 2} = \frac{(x-2)(x+2)}{(x-2)(x-1)} = \frac{x+2}{x-1}$.

(a) $\lim_{x \to 2^-} \frac{x+2}{x-1} = 4$
$\lim_{x \to 2^+} \frac{x+2}{x-1} = 4$
$\lim_{x \to 2} \frac{x+2}{x-1} = \boxed{4}$

(b) $\lim_{x \to -2^-} \frac{x+2}{x-1} = 0$
$\lim_{x \to -2^+} \frac{x+2}{x-1} = 0$
$\lim_{x \to -2} \frac{x+2}{x-1} = \boxed{0}$

(c) $\lim_{x \to 1^-} \frac{x+2}{x-1} = -\infty$
$\lim_{x \to 1^+} \frac{x+2}{x-1} = \infty$
$\lim_{x \to 1} \frac{x+2}{x-1}$ $\boxed{\text{DNE}}$

Ex. 8 Consider $f(x) = \frac{\frac{1}{2+x} - \frac{1}{2}}{x} = \frac{\frac{2}{2(2+x)} - \frac{2+x}{2(2+x)}}{x}$

$= \frac{2-(2+x)}{2(2+x)} \cdot \frac{1}{x} = \frac{-x}{2(2+x)} \cdot \frac{1}{x}$

$= -\frac{1}{2(2+x)}$

$\lim_{x \to 0} f(x) = \lim_{x \to 0} \left(-\frac{1}{2(2+x)}\right)$

$= -\frac{1}{2(2+0)} = \boxed{-\frac{1}{4}}$

Ex.9 Consider $f(x) = \dfrac{\frac{3}{x+4} - \frac{3}{4}}{x} = \dfrac{1}{x} \cdot \dfrac{3(4) - 3(x+4)}{4(x+4)}$

$= \dfrac{12 - 3x - 12}{4x(x+4)} = -\dfrac{3}{4(x+4)}$

$\lim_{x \to 0} f(x) = \lim_{x \to 0} \left(-\dfrac{3}{4(x+4)}\right) = -\dfrac{3}{4(0+4)} = \boxed{-\dfrac{3}{16}}$

Ex.10 Find $\lim_{x \to 0} \dfrac{\sqrt{x+3} - \sqrt{3}}{x}$. $\left(= \dfrac{\sqrt{3} - \sqrt{3}}{0} = \dfrac{0}{0}\right)$

Rationalize!

$\dfrac{\sqrt{x+3} - \sqrt{3}}{x} \cdot \dfrac{\sqrt{x+3} + \sqrt{3}}{\sqrt{x+3} + \sqrt{3}} = \dfrac{(\sqrt{x+3})^2 - (\sqrt{3})^2}{x(\sqrt{x+3} + \sqrt{3})}$

$= \dfrac{x + 3 - 3}{x(\sqrt{x+3} + \sqrt{3})}$

$= \dfrac{1}{\sqrt{x+3} + \sqrt{3}}$

$\lim_{x \to 0} \left(\dfrac{1}{\sqrt{x+3} + \sqrt{3}}\right) = \dfrac{1}{\sqrt{3} + \sqrt{3}} = \boxed{\dfrac{1}{2\sqrt{3}}}$

Ex.11 Find $\lim_{x \to 0} \dfrac{\sqrt{2x+3} - \sqrt{3}}{x}$. $\left(= \dfrac{\sqrt{3} - \sqrt{3}}{0} = \dfrac{0}{0}\right)$

Rationalize!

$\dfrac{\sqrt{2x+3} - \sqrt{3}}{x} \cdot \dfrac{\sqrt{2x+3} + \sqrt{3}}{\sqrt{2x+3} + \sqrt{3}} = \dfrac{2x + 3 - (3)}{x(\sqrt{2x+3} + \sqrt{3})} = \dfrac{2}{\sqrt{2x+3} + \sqrt{3}}$

$\lim_{x \to 0} \dfrac{2}{\sqrt{2x+3} + \sqrt{3}} = \dfrac{2}{\sqrt{3} + \sqrt{3}} = \dfrac{2}{2\sqrt{3}} = \boxed{\dfrac{1}{\sqrt{3}}}$

Note: For piecewise functions, check if there is a different function directly to the left or right of the point.

Ex.12 $f(x) = \begin{cases} x^2, & x \le -2 \\ 2x+1, & -2 < x \le 0 \\ 1 - x, & x > 0 \end{cases}$

← Use this function for values to the left of -2.

← Use this function for values to the right of -2 and the left of 0.

Use this for values to the right of 0.

(a) $\lim_{x \to -2} f(x) = ?$

$\lim_{x \to -2^-} f(x) = \lim_{x \to -2^-} (x^2) = (-2)^2 = 4$

$\lim_{x \to -2^+} f(x) = \lim_{x \to -2^+} (2x+1) = 2(-2)+1 = -3$

$\lim_{x \to -2^-} f(x) \neq \lim_{x \to -2^+} f(x)$, so $\lim_{x \to -2} f(x)$ $\boxed{\text{DNE}}$

(b) $\lim_{x\to 0} f(x) = ?$

$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} (2x+1) = 2(0)+1 = 1$

$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} (1-x) = 1-0 = 1$

$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x) = 1$, so $\lim_{x\to 0} f(x) = \boxed{1}$

(c) $\lim_{x\to -1} f(x) = ?$

-1 is not an endpoint of the intervals for the piecewise function, so

$\lim_{x\to -1} f(x) = \lim_{x\to -1} (2x+1) = 2(-1)+1 = \boxed{-1}$

Ex. 13 $f(x) = \begin{cases} x-1 & , x<1 \\ 1 & , x=1 \\ x^2-1 & , x>1 \end{cases}$

(a) $f(1) = ?$

$f(1) = 1$

(b) $\lim_{x\to 1} f(x) = ?$

$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} (x-1) = 1-1 = 0$

$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} (x^2-1) = 1^2-1 = 0$

$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x) = 0$, so $\lim_{x\to 1} f(x) = \boxed{0}$

(even though $f(1)=1$).

Ex. 14 $f(x) = \begin{cases} 2x+7 & , x\neq 0 \\ 17 & , x=0 \end{cases}$

Find $\lim_{x\to 0} f(x)$.

The only place the function is not $2x+7$ is when $x=0$, so the left and right limits both use $2x+7$. Then

$\lim_{x\to 0} f(x) = \lim_{x\to 0} (2x+7) = 2(0)+7 = \boxed{7}$.

Ex. 15 $f(x) = \begin{cases} 3x & , \ x < \frac{\pi}{6} \\ \pi\sin(x) & ; \ \frac{\pi}{6} \le x < \frac{\pi}{3} \\ \cos(x)+1 & , \ x \ge \frac{\pi}{3} \end{cases}$

(a) $\lim_{x \to \frac{\pi}{6}} f(x) = ?$

$\lim_{x \to \frac{\pi}{6}^-} f(x) = \lim_{x \to \frac{\pi}{6}^-} (3x) = 3\left(\frac{\pi}{6}\right) = \frac{\pi}{2}$

$\lim_{x \to \frac{\pi}{6}^+} f(x) = \lim_{x \to \frac{\pi}{6}^+} (\pi\sin(x)) = \pi\sin\left(\frac{\pi}{6}\right) = \pi\left(\frac{1}{2}\right) = \frac{\pi}{2}$

Then $\lim_{x \to \frac{\pi}{6}^-} f(x) = \lim_{x \to \frac{\pi}{6}^+} f(x) = \frac{\pi}{2}$,

so $\lim_{x \to \frac{\pi}{6}} f(x) = \boxed{\frac{\pi}{2}}$

(b) $\lim_{x \to \frac{\pi}{3}} f(x) = ?$

$\lim_{x \to \frac{\pi}{3}^-} f(x) = \lim_{x \to \frac{\pi}{3}^-} (\pi\sin(x)) = \pi\sin\left(\frac{\pi}{3}\right) = \pi\frac{\sqrt{3}}{2} = \frac{\pi\sqrt{3}}{2}$

$\lim_{x \to \frac{\pi}{3}^+} f(x) = \lim_{x \to \frac{\pi}{3}^+} (\cos(x)+1) = \cos\left(\frac{\pi}{3}\right)+1 = \frac{1}{2}+1 = \frac{3}{2}$.

Then $\lim_{x \to \frac{\pi}{3}^-} f(x) \neq \lim_{x \to \frac{\pi}{3}^+} f(x)$,

so $\lim_{x \to \frac{\pi}{3}} f(x)$ $\boxed{\text{DNE}}$