Exam 2 Review

Lessons 11 - 12: The Chain Rule and Derivative of Natural Log

The derivative of the outside (with the inside plugged back in) times the derivative of the inside.

* Special Case: \( \frac{d}{dx} \left( e^{f(x)} \right) = f'(x) \cdot e^{f(x)} \) (Note: the exponent does NOT change.)

* \( \frac{d}{dx} \left( \ln(x) \right) = \frac{1}{x} \)

* Use rules for logs and exponents to potentially make computing derivatives easier.

\[
\begin{align*}
\ln(xy) &= \ln(x) + \ln(y) & e^x e^y &= e^{x+y} & x^a x^b &= x^{a+b} \\
\ln\left(\frac{x}{y}\right) &= \ln(x) - \ln(y) & \frac{e^x}{e^y} &= e^{x-y} & \frac{x^a}{x^b} &= x^{a-b} \\
\ln(x^y) &= y \ln(x) & \frac{1}{e^x} &= e^{-x} & b \sqrt{x^a} &= x^\frac{a}{b} \\
x^{-a} &= \frac{1}{x^a} & (e^x)^y &= e^{xy} & (x^a)^b &= x^{ab}
\end{align*}
\]

Lesson 13: Higher Order Derivatives

* \( f^{(n)}(x), \frac{d^n y}{dx^n}, y^{(n)} \) all mean to take the derivative \( n \) times.

* If \( s(t) \) is a position function, \( s'(t) = v(t) \) is velocity, and \( s''(t) = v'(t) = a(t) \) is acceleration.
**Lesson 14: Implicit Differentiation**

<table>
<thead>
<tr>
<th>Implicit Differentiation</th>
</tr>
</thead>
</table>
| \[
\frac{d}{dx}[f(y)] = \frac{dy}{dx} \cdot f'(y)
\]|

⁻ Use when \( y \) is not explicitly solved for. For instance: \( y^2 = e^x, \cos(xy) = x \), etc.

⁻ Basically, anytime we “touch/change” a \( y \), we need to multiply \( \frac{dy}{dx} \) or \( y' \) to the term, so take the derivative of what you see and put the \( \frac{dy}{dx} \) or \( y' \) next to it.

⁻ For example,

\[
\frac{d}{dx}(e^y) = \frac{dy}{dx} \cdot e^y
\]

\[
\frac{d}{dx}(\ln(y)) = \frac{dy}{dx} \cdot \frac{1}{y}
\]

\[
\frac{d}{dx}(x \cdot y) = 1 \cdot y + x \cdot \frac{dy}{dx} \quad \text{(product rule)}
\]

\[
\frac{d}{dx}(e^{xy}) = \left(y + x \cdot \frac{dy}{dx}\right)e^{xy} \quad \text{(chain rule and product rule)}
\]

\[
\frac{d}{dx}(y^2) = 2y \cdot \frac{dy}{dx}
\]

⁻ Your \( \frac{dy}{dx} \) or \( y' \) will **not** appear inside a trig function, denominator, or exponent.

**Lessons 15-16: Related Rates**

⁻ Determine what formula you need to use.

⁻ Plug in any **constant** quantities, i.e. quantities that do **not** change with **time**.

⁻ Take derivative of both sides with respect to **time**.

⁻ Pay attention to units!

<table>
<thead>
<tr>
<th>Shape</th>
<th>Perimeter ( P ) or Surface Area ( S )</th>
<th>Area ( A ) or Volume ( V )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rectangle with sides ( l ) and ( w )</td>
<td>( P = 2l + 2w )</td>
<td>( A = lw )</td>
</tr>
<tr>
<td>Square with side ( x )</td>
<td>( P = 4x )</td>
<td>( A = x^2 )</td>
</tr>
</tbody>
</table>
| Circle with radius \( r \) | \( (\text{Circumference } C) \)
\( C = 2\pi r \) | \( A = \pi r^2 \) |
<p>| Cube with side ( x ) | ( S = 6x^2 ) | ( V = x^3 ) |</p>
<table>
<thead>
<tr>
<th>Trapezoid</th>
<th>(with sides $a, b, c, d$) $P = a + b + c + d$</th>
<th>(with base $b_1$ and $b_2$ and height $h$) $A = \frac{1}{2} h (b_1 + b_2)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rectangular prism with sides $l, w, h$</td>
<td>$S = 2lw + 2lh + 2wh$</td>
<td>$V = lwh$</td>
</tr>
<tr>
<td>Triangle</td>
<td>(with sides $a, b, c$) $P = a + b + c$</td>
<td>(with base $b$ and height $h$) $A = \frac{1}{2}bh$</td>
</tr>
</tbody>
</table>

- For right triangles, we use the Pythagorean Theorem $x^2 + y^2 = D^2$.
- For angles, pick the trig identity that has a constant quantity and the rate of the other quantity given.

**Lessons 17-18:** Relative Extrema, Critical Numbers, and the First Derivative Test

- We can use the *first derivative* to find out information about the function itself.
- **Critical Numbers:** where the derivative of a function $(f'(x))$ is $= 0$ or is *undefined*.
- A critical number must be in the domain of the function!
- **Relative Extrema:** minimums or maximums of the entire function; function must be defined at the point; can have no relative extrema or can have multiple relative minimums or maximums

- $f(x)$ is increasing when $f'(x) > 0$
- $f(x)$ is decreasing when $f'(x) < 0$

- Relative extrema only occur at critical numbers, but not all critical numbers have relative extrema.
  - We find the critical numbers (always $x$-values) then check at each $x$ coordinate to see if we have a relative minimum, a relative maximum or neither.

- **First Derivative Test:** Let $c$ be a critical number for $f(x)$.
  - If $f'(x)$ goes from positive to negative at $x = c$, (i.e., $f(x)$ goes from increasing to decreasing),
    
    $f(c)$ is a **relative maximum**.
  - If $f'(x)$ goes from negative to positive at $x = c$, (i.e., $f(x)$ goes from decreasing to increasing),
    
    $f(c)$ is a **relative minimum**.
  - If $f'(x)$ does not change sign at $x = c$, $f(c)$ is neither a min nor a max.

- Use the original function to find the value of the minimum or maximum, i.e. $f(c)$.

- **Notation:** $x = c$ is where the relative min or max occurs; $f(c)$ is the value of the minimum or maximum.
  - If they ask you to find the minimum or maximum, they want you to find the value of the function at that point.
### Table of Derivatives

#### Derivatives of specific functions

- \( \frac{d}{dx}(c) = 0 \)
- \( \frac{d}{dx}(x^n) = n \cdot x^{n-1} \)
- \( \frac{d}{dx}(e^x) = e^x \)
- \( \frac{d}{dx}(\sin(x)) = \cos(x) \)
- \( \frac{d}{dx}(\tan(x)) = \sec^2(x) \)
- \( \frac{d}{dx}(\sec(x)) = \sec(x) \cdot \tan(x) \)
- \( \frac{d}{dx}(\cos(x)) = -\sin(x) \)
- \( \frac{d}{dx}(\cot(x)) = -\csc^2(x) \)
- \( \frac{d}{dx}(\csc(x)) = -\csc(x) \cdot \cot(x) \)

#### Differentiation Rules

- \( \frac{d}{dx}(c \cdot f(x)) = c \cdot f'(x) \)
- \( \frac{d}{dx}(f(x) + g(x)) = f'(x) + g'(x) \)
- \( \frac{d}{dx}(f(x) - g(x)) = f'(x) - g'(x) \)
- \( \frac{d}{dx}(f(x) \cdot g(x)) = f'(x)g(x) + f(x)g'(x) \)
- \( \frac{d}{dx}(\frac{f(x)}{g(x)}) = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2} \quad \text{OR} \quad \frac{d}{dx}(\frac{\text{Top}}{\text{Bottom}}) = \frac{\text{Top}' \cdot \text{Bottom} - \text{Top} \cdot \text{Bottom}'}{(\text{Bottom})^2} \)

#### New Derivative Types

- **Chain Rule**
  \( \frac{d}{dx}[\text{Out}(\ln)] = \text{Out}'(\ln) \cdot \ln' \)

- **Implicit Differentiation**
  \( \frac{d}{dx}[f(y)] = f'(y) \cdot \frac{dy}{dx} \)

- \( \frac{d}{dx}[e^{f(x)}] = f'(x) \cdot e^{f(x)} \)

- \( \frac{d}{dt}[x + y] = \frac{dx}{dt} + \frac{dy}{dt} \)

- \( \frac{d}{dx}\ln(f(x)) = \frac{f'(x)}{f(x)} \)