Final Exam Review

Exam 1 Material

Lessons 2 - 4: Limits

* Limit Solving Strategy for $\lim_{x \to c} f(x) = ?$

For *piecewise functions*, you always need to look at the left and right limits!

If f(x) is not a piecewise function, plug c into f(x), i.e., find f(c).

There are three cases that will occur:

Case 1: f(c) = finite number, then $\lim_{x \to c} f(x) = f(c)$

Case 2: $f(c) = \frac{\text{nonzero number}}{0}$, then look at left and right limits to decide if the limit is $\infty, -\infty, DNE$. Case 3: $f(c) = \frac{0}{0}$, then manipulate f(x) to get case 1 or case 2. Usually you need to factor and simplify f(x), or if you have square roots, you need to rationalize f(x) by multiplying the numerator and denominator by the conjugate – remember $a^2 - b^2 = (a - b)(a + b)$.

- * One-Sided Limits (need to consider for *Case 2* above and for *piecewise functions*, but you may be explicitly asked to find them as well.)
 - Left Limit: $\lim_{x \to c^-} f(x)$ for x < c
 - Right Limit: $\lim_{x \to c^+} f(x)$ for x > c
- * Remember that the limit $\lim_{x \to c} f(x)$ exists if $\lim_{x \to c^-} f(x) = \lim_{x \to c^+} f(x)$

* A limit can exist even if the function is undefined! If a function is undefined, or the value of the function does not equal the limit there, those are issues of *continuity*.

* If you're still unsure what the limit is, you can find it **numerically** or **graphically**.

To find the limit **numerically**, pick values really close to *c*. For example, if c = 1, pick 0.9999 (for the left limit since 0.9999 < 1) and 1.0001 (for the right limit since 1.0001 > 1) and plug them into f(x). Finding the limit **graphically** will be difficult unless you already have a good idea how f(x) looks or if you are given the graph of f(x)

Practice Problems: Exam 1 - # 1, 3, 4 Exam 1 Review - # 1, 2, 4, 6, 8, 9, 12, 13, 14, 17, 18, 20, 30, 31

Lesson 5: Continuity

- # f(x) is **continuous** at x = c if ALL of the following 3 things are true:
 - ① f(c) is defined
 - ⁽²⁾ $\lim_{x \to c} f(x)$ exists
 - $Iim_{x \to c} f(x) = f(c)$
- * Types of Discontinuity (which conditions above hold at the particular x-value?)

Classification \rightarrow Condition \checkmark	Hole	Jump	Vertical Asymptote
1	Maybe	Maybe	No
2	Yes (finite)	No (but both are finite)	Maybe (∞ , $-\infty$, or DNE)
3	No	No	No

- **Hole:** Happen when we cancel out a factor in a rational function. (Ex. $f(x) = \frac{(x+1)(x-1)}{x-1} = x + 1$ has a hole at x = 1 because we canceled out the x 1 factor from the denominator.) They can also happen in piecewise functions when the limit exists, but the function is not defined there or if the function value does not equal the limit.
- * Jump: Happen only in piecewise functions when the left and right limits are *finite* and do not match.
- * Vertical Asymptote: Happen when we can't cancel out a factor in the denominator. (Ex. $f(x) = \frac{x-1}{(x-1)(x+1)} = \frac{1}{x+1}$ has a vertical asymptote at x = -1 because we can't cancel out the x + 1 factor. Notice that this function also has a hole at x = 1 because we can cancel the x - 1 factor.)
- For piecewise functions, check the *x*-values where the different functions meet to see if the limit exists and/or if the limit equals the value of the function to find holes or jumps.
 For non-piecewise functions, find where the denominator equals 0. Factor the numerator and denominator. If a factor cancels from the denominator, there will be a hole. If a factor does not cancel from the denominator, there will be a vertical asymptote.

Practice Problems: Exam 1 - # 2, 3 Exam 1 Review - # 2, 3, 8, 9, 17, 19, 32

Lesson 6: Limit Definition of the Derivative

- * The **derivative** of a function is the <u>slope</u> of the tangent line.
- * Limit Definition of Derivative: $f'(x) = \lim_{h \to 0} \frac{f(x+h) f(x)}{h}$
- * We need to keep in mind *case 3* for finding a limit because generally, we end up with $\frac{0}{0}$ when trying to use the limit definition for a derivative. You may need to expand the numerator or denominator, rationalize, factor, etc. to get the *h* in the denominator to cancel with an *h* in the numerator.

Practice Problems: Exam 1 - # 5 Exam 1 Review - # 7, 15, 33

Lessons 7 - 10: Instantaneous Rates of Change, Differentiation Rules and Trig Functions

- * When you're given a function, you take the derivative to find the **rate of change** of the function.
- * Pay attention to units!
- * Remember if s(t) is a **position** function, s'(t) = v(t) the **velocity** function and v'(t) = a(t) the **acceleration** function
- * Pay close attention to operations, i.e., adding, subtracting, multiplying, or dividing. If functions are multiplied (Ex: $f(x) = e^x \sin(x)$, $f(x) = x^2 e^x$, etc), use the product rule to take the derivative.

If functions are divided (Ex: $f(x) = \frac{x}{x^2+1}$, $f(x) = \frac{e^x}{\sin(x)}$, etc), use the quotient rule to take the derivative.

Practice Problems: Exam 1 - # 6, 7, 8, 9, 10, 11, 12

Exam 1 Review - # 5, 10, 11, 16, 21, 22, 23, 24, 25, 26, 27, 28, 29, 34, 35, 36, 37, 38, 39, 40

Exam 2 Material

Lessons 11 - 12: The Chain Rule and Derivative of Natural Log

* The derivative of the *outside* (with the *inside* plugged into the derivative) times the

derivative of the *inside*.

* Special Case: $\frac{d}{dx}(e^{f(x)}) = f'(x) * e^{f(x)}$ (Note: the exponent does NOT change.)

Practice Problems: Exam 2 - # 3, 4, 10 Exam 2 Review - # 6, 9, 10, 11, 12, 13, 18, 19, 29, 33

Lesson 13: Higher Order Derivatives

* $f^{(n)}(x)$, $\frac{d^n y}{dx^{n'}} y^{(n)}$ all mean to take the derivative *n* times.

Practice Problems: Exam 2 - # 1, 7 Exam 2 Review - # 1, 2, 22, 24, 25, 30, 38

Lesson 14: Implicit Differentiation

$$* \frac{d}{dx}[f(y)] = \frac{dy}{dx} * f'(y)$$

- * Use when y is not explicitly solved for. For instance: $y^2 = e^x$, $\cos(xy) = x$, etc.
- * Basically, anytime we "touch/change" a y, we need to multiply the term by $\frac{dy}{dx}$ or y', so take the derivative of what you see and multiply by $\frac{dy}{dx}$ or y'.
- ✤ For example,

$$\frac{d}{dx}(e^{y}) = \frac{dy}{dx} * e^{y}$$
$$\frac{d}{dx}(\ln(y)) = \frac{dy}{dx} * \frac{1}{y}$$

$$\frac{d}{dx}(x * y) = 1 * y + x * \frac{dy}{dx} \text{ (product rule)}$$

$$\frac{d}{dx}(e^{xy}) = \left(y + x\frac{dy}{dx}\right)e^{xy} \text{ (chain rule and product rule)}$$

$$\frac{d}{dx}(y^2) = \frac{dy}{dx} 2y$$

$$\text{ * } \frac{dy}{dx} \text{ and } y' \text{ will not appear inside a trig function, denominator, or exponent.}$$

Practice Problems: Exam 2 - # 6, 11 Exam 2 Review - # 3, 14, 20, 21, 32

Lessons 15-16: Related Rates

- * Determine what formula you need to use.
- * Plug in any *constant* quantities, i.e. quantities that do NOT change with *time*.
- * Take derivative of both sides with respect to *time*.
- * Pay attention to units!
- * For right triangles, we use the Pythagorean Theorem $x^2 + y^2 = D^2$.
- * For angles, pick the trig identity that has a *constant* quantity and the *rate* of the other quantity given.

Practice Problems: Exam 2 - # 2, 12 Exam 2 Review - # 4, 8, 15, 16, 17, 23, 28, 36, 37

Lessons 17-18: Relative Extrema, Critical Numbers, and the First Derivative Test

- * We can use the *first derivative* to find out information about the *function* itself.
- * Critical Numbers: where the derivative of a function (f'(x)) is = 0 or is *undefined*.
- * A critical number must be in the *domain* of the function!
- * **Relative Extrema:** minimums or maximums of the entire function; function must be defined at the point; can have no relative extrema or can have multiple relative minima or relative maxima
- * f(x) is increasing when f'(x) > 0
- * f(x) is **decreasing** when f'(x) < 0

- * Relative extrema **only** occur at critical numbers, but not *all* critical numbers have relative extrema.
 - We *find* the critical numbers (always *x*-values) then use the First or Second Derivative Test to *check* at each *x*-coordinate to see if we have a relative minimum, a relative maximum or neither.
- * First Derivative Test: Let c be a critical number for f(x), i.e. f'(c) = 0.
 - If f'(x) goes from positive to negative at x = c, (i.e., f(x) goes from increasing to decreasing), then f(c) is a *relative maximum*.
 - If f'(x) goes from negative to positive at x = c, (i.e., f(x) goes from decreasing to increasing), then f(c) is a *relative minimum*.
 - If f'(x) does not change sign at x = c, f(c) is neither a min nor a max.
- * Use the original function to find the value of the minimum or maximum, i.e. f(c).
- * Notation: x = c is where the relative min or max occurs; f(c) is the value of the minimum or maximum. If they ask you to find the minimum or maximum, they want you to find the value of the function at that point.

Practice Problems: Exam 2 - # 5, 8, 9 Exam 2 Review - # 5, 7, 26, 27, 31, 34, 35

Exam 3 Material

Lesson 19: Concavity, Inflection Points, and the Second Derivative Test

- * f(x) is concave up when f''(x) > 0
- * f(x) is concave down when f''(x) < 0
- * An **inflection point** is where f(x) changes concavity. To find the y-coordinate, plug the x-value into the original function f(x).
- * Second Derivative Test: Let c be a critical number for f(x) (i.e. f'(c) = 0).
 - If f''(c) > 0, f(x) is concave up at x = c, so f(c) is a relative minimum.
 - If f''(c) < 0, f(x) is concave down at x = c, so f(c) is a relative maximum.
 - If f''(c) = 0, the test is *inconclusive*, so use the first derivative test to classify the critical number.

Practice Problems: Exam 3 - # 1 Exam 3 Review - # 1, 2, 11, 16, 23, 24, 33, 38, 39

Lesson 20: Absolute Extrema on an Interval

How to find the *absolute* extrema of f(x) on a closed interval [a, b]:

- ① Find all critical numbers of f(x) (where f'(x) = 0).
- ⁽²⁾ Evaluate f(x) at all critical numbers in [a, b], and at x = a and x = b.
- ⁽³⁾ Compare the f(x) values and identify the absolute min and absolute max.

Practice Problems: Exam 3 - # 2 Exam 3 Review - # 12, 20, 36

Lesson 21: Graphical Interpretations of Derivatives

When given a graph of f'(x), we can find where f(x) is/has the following:

- 1. critical numbers where f'(x) = 0 or does not exist, i.e. where the graph of f'(x) touches the *x*-axis.
- 2. increasing where f'(x) > 0, i.e where the graph of f'(x) is *above* the x-axis.
- 3. decreasing where f'(x) < 0, i.e where the graph of f'(x) is below the x-axis.
- 4. relative maximum where the graph of f'(x) goes from *positive* to *negative*.

- 5. relative minimum where the graph of f'(x) goes from *negative* to *positive*.
- 6. concave up where f''(x) > 0, i.e. where f'(x) is increasing
- 7. concave down where f''(x) < 0, i.e. where f'(x) is *decreasing*
- 8. inflection point f''(x) changes sign, i.e. f'(x) has a *horizontal* tangent line and switches from increasing to decreasing or decreasing to increasing

Practice Problems: Exam 3 - # 5 Exam 3 Review - # 4, 40

Lesson 22: Limits at Infinity

When evaluating a limit with $x \to \infty$ or $x \to -\infty$, take the highest x power in the numerator and denominator with their coefficients, cancel, and then evaluate.

- * Asymptotes (for f(x))
 - Vertical: Simplify f(x) and set the denominator equal to 0 and solve for x. (x = #)
 - **Horizontal:** Find $\lim_{x \to \infty} f(x)$ and $\lim_{x \to -\infty} f(x)$. If one of the limits equals a finite L, y = L is the HA. If the limits are infinite $(-\infty \text{ or } \infty)$, there is no HA.
 - Slant: Occur when the degree of numerator is exactly one more than degree of the denominator. Find by *polynomial division*. (Of the form y = mx + b.)

Practice Problems: Exam 3 - # 3, 4, 8, Exam 3 Review - # 5, 6, 13, 14, 26, 27, 35, 37

Lesson 23: Curve Sketching

* To sketch a curve, find the following:

- 1. *x*-intercepts (when y = 0)
- 2. *y*-intercept (when x = 0)
- 3. Increasing/Decreasing Intervals (use *first derivative*)
- 4. Concave Up/Concave Down Intervals (use second derivative)
- 5. Inflection Points (use *second derivative*)
- 6. Vertical Asymptotes
- 7. Horizontal Asymptotes
- 8. Slant Asymptotes
- * When making number lines for the first and second derivatives, be sure to mark where the function is **undefined**, check the sign (positive or negative) on both sides of the discontinuity. None of the intervals should include the *x*-values where f(x) is undefined!

Practice Problems: Exam 3 - # 8 Exam 3 Review - # 3, 25

Lessons 24-26: Optimization

Draw a picture!

- 1. Identify the quantity to be optimized (maximized or minimized).
- 2. Identify the *constraint(s)*.
- Solve the constraint for one of the variables.
 Substitute this into the equation from Step 1.
- 4. Take the *derivative* with respect to the variable.
- 5. Set the derivative equal to zero and solve for the variable.
- 6. Find the desired quantity.

Note: In most cases, you only get one plausible solution. However, you can use the **1**st**DT** or **2**nd**DT** to check if your answer in step 5 is the critical number where the minimum or maximum occurs.

Practice Problems: Exam 3 - # 10, 11, 12 Exam 3 Review - # 7, 8, 9, 10, 15, 19, 21, 22, 28, 29, 30, 44, 45

Lessons 27-28: Antiderivatives and Indefinite Integration

- Rewrite the function, if necessary, to utilize the table of integration.
- We have NOT learned methods to "undo" the *chain, product*, or *quotient* rules.
- Once you have integrated, you can take the derivative to double check your integration.
- When given an initial value or initial values, you can use them to find the *particular solution* (i.e. you can solve for *C*).

Practice Problems: Exam 3 - # 6, 7, 9 Exam 3 Review - # 17, 18, 31, 32, 34, 41, 42, 43

Post Exam 3 Material

Lesson 29: Area and Riemann Sums

***** Summations:

Ex.
$$\sum_{i=2}^{5} (i+i^2) = (2+2^2) + (3+3^2) + (4+4^2) + (5+5^2) = (6) + (12) + (20) + (30) = 68$$

* Estimate the area under the curve f(x) with *n* rectangles on the interval [a, b].

Left Riemann Sum:

$$\sum_{i=0}^{n-1} f(x_i) * \Delta x$$

Right Riemann Sum:

$$\sum_{i=1}^n f(x_i) * \Delta x$$

where $\Delta x = \frac{b-a}{n}$

$$x_i = a + i * \Delta x$$
, $i = 0, 1, 2, 3, ..., n$

After Exam 3 Review - # 4, 13, 24, 30, 34, 35, 45, 46, 49

Lesson 30: Definite Integrals

- * $\int_{a}^{b} f(x) dx$ is the area under the curve f(x) (between f(x) and the x-axis) on the interval [a, b].
- * Pay attention to coefficients in given integrals!!
- * Properties of Definite Integrals:

$$\circ \int_{a}^{a} f(x)dx = 0$$

$$\circ \int_{a}^{b} f(x)dx = -\int_{b}^{a} f(x)dx$$

$$\circ \int_{a}^{b} k f(x)dx = k \int_{a}^{b} f(x)dx$$

$$\circ \int_{a}^{b} (f(x) \pm g(x)) dx = \int_{a}^{b} f(x)dx \pm \int_{a}^{b} g(x)dx$$

$$\circ \int_{a}^{c} f(x)dx = \int_{a}^{b} f(x)dx + \int_{b}^{c} f(x)dx$$

After Exam 3 Review - # 5, 23, 47, 48

Lessons 31-32: The Fundamental Theorem of Calculus

- # If $\int f'(x)dx = f(x)$, then $\int_a^b f'(x)dx = f(b) f(a)$.
- * When we are given an initial value, we can find the particular solution for indefinite integration (lessons 27-28). When we do NOT have an initial value, we can find the change in the function between two points a and b by using the fundamental theorem of calculus.

After Exam 3 Review - # 3, 6, 7, 8, 9, 14, 15, 16, 21, 22, 25, 26, 27, 36, 37, 38, 39, 40, 41, 44

Lesson 33: Numerical Integration

* Approximate the area under f(x) on [a, b] with n trapezoids:

$$T_n = \frac{1}{2} * \Delta x * [f(x_0) + 2f(x_1) + 2f(x_2) + \dots + 2f(x_{n-1}) + f(x_n)]$$

* where $\Delta x = \frac{b-a}{n}$ $x_i = a + i * \Delta x$, i = 0, 1, 2, 3, ..., n

After Exam 3 Review - # 10, 28, 42, 49

Lessons 34/36: Exponential Growth/Decay

- * If the rate of change of y is proportional to y, i.e. $\frac{dy}{dt} = ky$ where k is the growth/decay rate, then $y(t) = Ce^{kt}$
- * C = y(0)
- * If k is negative, we have exponential decay.
 If k is positive, we have exponential growth.
- * For savings accounts with interest *compounded continuously*, we have $A(t) = Pe^{rt}$, where r is the interest rate as a decimal, P is the principle (original amount invested), and t is time in years.
- * The half-life $t_{1/2}$ for exponential/radioactive decay is the time it takes for half of an amount to decay.

$$k = \frac{\ln\left(\frac{1}{2}\right)}{t_{1/2}}$$

After Exam 3 Review - # 1, 2, 11, 12, 17, 18, 19, 20, 29, 31, 32, 33, 43

Table of Derivatives

$$\frac{d}{dx}(c) = 0$$

$$\frac{d}{dx}(x^{n}) = n x^{n-1}, \quad n \neq 0$$

$$\frac{d}{dx}(x^{n}) = n x^{n-1}, \quad n \neq 0$$

$$\frac{d}{dx}(x^{n}) = n x^{n-1}, \quad n \neq 0$$

$$\frac{d}{dx}(x^{n}) = x^{n-1}, \quad x > 0$$

$$\frac{d}{dx} \left[e^{f(x)} \right] = f'(x) \, e^{f(x)}$$

Table of Integration

$$\int 0 \, dx = C$$

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1} + C, \quad n \neq -1$$

$$\int k \, dx = kx + C$$

$$\int e^x \, dx = e^x + C$$

$$\int \int e^x \, dx = e^x + C$$

$$\int \int e^x \, dx = \ln|x| + C$$

$$\int \cos(x) \, dx = \sin(x) + C$$

$$\int \sin(x) \, dx = -\cos(x) + C$$

$$\int \sin(x) \, dx = -\cos(x) + C$$

$$\int \sec^2(x) \, dx = \tan(x) + C$$

$$\int \sec(x) \tan(x) \, dx = \sec(x) + C$$

$$\int \csc^2(x) \, dx = -\cot(x) + C$$

$$\int \csc^2(x) \, dx = -\cot(x) + C$$

$$\int \csc(x) \cot(x) \, dx = -\csc(x) + C$$

$$\int cf'(x) \, dx = c \int f'(x) \, dx$$

$$\int (f'(x) \pm g'(x)) \, dx = \int f'(x) \, dx \pm \int g'(x) \, dx$$

We don't have a way to undo products or quotients except for the specific trig functions above. We also cannot undo the chain rule yet. To integrate things that look like products, quotients, or chains, we need to

rewrite them!