Some Specific Comments

✳ Limits

Piecewise Functions: When $f(x)$ is a piecewise function, and c is the endpoint for one of the pieces, we MUST use one-sided limits to find the overall limit.

Remember that $\lim_{x\to c} f(x)$ exists only if $\lim_{x\to c^-} f(x) = \lim_{x\to c^+} f(x)$.

Ex. Given $f(x) = \begin{cases} -x, & x < 1 \\ x, & x \geq 1 \end{cases}$, find $\lim_{x\to 1} f(x)$.

To find $\lim_{x\to 1^-} f(x)$, we look at x values with $x < 1$. Looking at the definition of $f(x)$, when $x < 1$, we use the top part of the function $-x$, so we have

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} (-x) = -1.$$

To find $\lim_{x\to 1^+} f(x)$, we look at x values with $x > 1$. Looking at the definition of $f(x)$, when $x > 1$, we use the bottom part of the function x, so we have

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} (x) = 1.$$

This means that $\lim_{x\to 1^-} f(x) \neq \lim_{x\to 1^+} f(x)$, so $\lim_{x\to 1} f(x)$ does not exist.

Limits NOT at Infinity: **Ex**. Given $f(x) = \frac{x+1}{x^2-1}$, find $\lim_{x\to 1} f(x)$ and $\lim_{x\to -1} f(x)$.

Let's look at $\lim_{x\to 1} f(x)$ first. The first thing we should do is find $f(1)$. Here, we get $f(1) = \frac{(1+1)}{1^2-1} = \frac{2}{0}$. Since we have $\frac{\text{nonzero number}}{\text{zero}}$, the denominator is getting really small, so the whole thing will get very big. This means the limit will be ∞, $-\infty$, or DNE. To decide, we look at the one-sided limits. When the one of them is ∞ and the other is $-\infty$, the limit DNE. Here,

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} \frac{x+1}{x^2-1} = \frac{2}{\text{negative } \# \to 0} = -\infty$$

since $x \to 1^-$ means $x < 1$, $x^2 - 1 < 0$, so the denominator is negative and goes to 0. Similarly,

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} \frac{x+1}{x^2-1} = \frac{2}{\text{positive } \# \to 0} = \infty$$

since $x \to 1^+$ means $x > 1$, $x^2 - 1 > 0$, so the denominator is positive and goes to 0.

Since the one-sided limits aren't equal, $\lim_{x\to 1} f(x)$ does not exist.

Now, let's look at $\lim_{x\to -1} f(x)$. Again, the first thing we should do is find $f(-1)$. Here, we get $f(-1) = \frac{-1+1}{(-1)^2-1} = \frac{0}{0}$, so we need to simplify the function. $f(x) = \frac{x+1}{(x-1)(x+1)} = \frac{1}{x-1}$. Now, we have $\lim_{x\to -1} f(x) = \lim_{x\to -1} \frac{1}{x-1} = \frac{1}{-1-1} = \frac{1}{-2} = -\frac{1}{2}$.

In general, the best idea is to just plug the number into the function, then refer to the cases in the Exam 1 Review for Lessons 2-4. When in doubt though, you can find limits numerically by plugging in x-values that are VERY close to the number. For instance, if you're trying to find the limit as x goes to -3, you should plug in numbers like -3.0001 and -2.9999 rather than numbers like -3.5 and -2.5.

Limits at Infinity: The difference between limits at infinity and limits not at infinity is that for limits AT infinity, the **x-values** are going to infinity. However, BOTH limits AT infinity and limits NOT at infinity can EQUAL infinity. The limit is what the **y-values** or function values are going. The "AT" means the **x-values** are going to infinity.

<u>**Ex**</u>. Given $f(x) = \frac{x^3}{x^2-2x+1}$, find $\lim_{x\to 1} f(x)$ and $\lim_{x\to\infty} f(x)$.

To find $\lim_{x\to 1} f(x)$, we plug $x = 1$ into the function and get $f(1) = \frac{1^3}{1^2-2(1)+1} = \frac{1}{0}$. If we check the left and right limits though, we get $\lim_{x\to 1} f(x) = \infty$. (Check for yourself!)

To find $\lim_{x\to\infty} f(x)$, we take the highest power terms in the numerator and denominator and think about what happens as x gets really big.

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{x^3}{x^2-2x+1} = \lim_{x\to\infty} \frac{x^3}{x^2} = \lim_{x\to\infty} x = \infty$$

So, both $\lim_{x\to 1} f(x)$ and $\lim_{x\to\infty} f(x)$ EQUAL infinity, but we use very different methods to evaluate them because $\lim_{x\to\infty} f(x)$ is a limit AT infinity and $\lim_{x\to 1} f(x)$ is not.

✳ **Rate of Change**

To find the rate of change of a quantity, we take the derivative.

When given the rate of change for a quantity, we integrate to find an expression for the quantity. If we're given an initial value, we can find the exact expression for the function. If we are not given an initial value, we can find the *change* in the quantity between two times or x values by using the Fundamental Theorem of Calculus.

✳ **Optimization versus Related Rates**

Optimization is when we are concerned with controlling different variables to maximize or minimize another quantity. When we optimize, we are concerned with the *constraints* given. (For instance, area must be 25 feet squared, perimeter must be 74 meters, volume must be 92 cubic feet, etc.)

For related rates, we are concerned about quantities changing with *time*. The most important part of related rates is do NOT plug in values that change with time until AFTER you've taken the derivative.

✳ **Find the equation of a tangent line to $f(x)$ at $x = c$:**

1. Find the slope, m, of the tangent line by taking the derivative and evaluating at the given x-value. Meaning, $m = f'(c)$.
2. Find a point on the line by plugging the given x-value into the *original* function. Meaning, find $f(c)$.
3. Now you have a point on the line and the slope of the line, so find $y = mx + b$.

   Method 1: Start with $y = mx + b$ and plug in m from Step 1 and $\big(c, f(c)\big)$ from Step 2. Then solve for b. Now that you know m and b, you can write $y = mx + b$.

   Method 2: Use point slope form $y - y_1 = m(x - x_1)$. Plug in m from Step 1 and $(x_1, y_1) = \big(c, f(c)\big)$ from Step 2. Then solve for y.

Math Tips for the Final

- Always look for product, quotient and chain rules when taking derivatives. Especially chain rule! Remember, if you see anything plugged into a function that's more complicated than x, even something like $-x$, you may have a chain rule.
- Try to rewrite functions to make taking derivatives or integrating easier.
- Simplify numbers/expressions as you work through a problem to avoid dealing with complicated expressions at the end.
- Store any long decimal values (especially for exponential growth/decay). Use the STO button to store in 1, 2, or 3, and the RCL button to recall the values. To clear a memory location, store 0 over the number you have stored.
- Draw a picture!
- Pay attention to units.
- Pay attention to parentheses! Specifically, do NOT insert them when they're not there, i.e. $-x^2 \neq (-x)^2$ and $\sin(x) + 1 \neq \sin(x + 1)$.
- Read problems carefully. Make sure you know if you're given $f(x)$, $f'(x)$, etc. Most importantly, read if you're given the graph of a function or the graph of the *derivative* of a function.
- Know what you're being asked to find, i.e., the x-value, the $f(x)$-value, the min, the max, etc.

General Tips for the Final

- Get plenty of sleep the night before the exam. Seriously. Don’t stay up late cramming. The best thing you can do is get a full night of sleep so that your mind is clear and focused the next day. It will be easier to recall information and interpret the questions on the exam.
- If you find yourself getting stressed during the exam, try closing your eyes and taking a deep breath to clear your mind.
- If you feel like you’re going in circles and not getting anywhere on a question, try erasing everything and starting again, or move on to another question and try that problem again later.
- Take your time! It’s better to work slowly and diligently on 20 questions and get them all correct than it is to rush to finish all 25 and risk making silly mistakes and missing more than 5 questions.
- Sorry for sounding like a life coach, but remember, it's just an exam. Getting frustrated and stressed on the exam will make it harder to focus, and you’ll start making mistakes.