Lesson 10: Quotient Rule; Other Trig Functions

Recall: $\frac{d}{dx}(\cos x) = -\sin x$

$\frac{d}{dx}(\sin x) = \cos x$

$\frac{d}{dx}(f(x)g(x)) = f'(x)g(x) + f(x)g'(x)$

* Quotient Rule: $\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}$

Ex.1 Differentiate $y = \frac{x^3+x}{x}$.

M1: $y = \frac{x^3+x}{x} = \frac{x^3}{x} + \frac{x}{x}$

$= x^2+1$

$\boxed{y' = 2x}$

M2: $f(x) = x^3+x \qquad g(x) = x$

$f'(x) = 3x^2+1 \qquad g'(x) = 1$

$$y' = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}$$

$$= \frac{(3x^2+1)(x) - (x^3+x)(1)}{(x)^2}$$

$$= \frac{3x^3+x-x^3-x}{x^2} = \frac{2x^3}{x^2}$$

$$\boxed{= 2x}$$

Ex.2 Differentiate $y = \frac{x^2+\sqrt{x}}{e^x}$.

$f(x) = x^2+\sqrt{x} \qquad g(x) = e^x$

$f'(x) = 2x + \frac{1}{2\sqrt{x}} \qquad g'(x) = e^x$

$$y' = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}$$

$$= \frac{(2x+\frac{1}{2\sqrt{x}})(e^x) - (x^2+\sqrt{x})(e^x)}{(e^x)^2}$$

$$= \frac{e^x(2x+\frac{1}{2\sqrt{x}} - x^2 - \sqrt{x})}{e^{2x}} = \frac{(2x+\frac{1}{2\sqrt{x}} - x^2 - \sqrt{x})}{e^x}$$

$$= \boxed{\frac{1}{e^x}\left(2x + \frac{1}{2\sqrt{x}} - x^2 - \sqrt{x}\right)}$$

$$\left(\text{or } e^{-x}\left(2x + \frac{1}{2\sqrt{x}} - x^2 - \sqrt{x}\right)\right)$$

Ex.3 Differentiate $y = \tan x$.

Recall: $y = \tan x = \frac{\sin x}{\cos x}$.

$f(x) = \sin x \qquad g(x) = \cos x$

$f'(x) = \cos x \qquad g'(x) = -\sin x$

$$y' = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}$$

$$= \frac{\cos x(\cos x) - (\sin x)(-\sin x)}{(\cos x)^2}$$

Recall: $\sin^2 x + \cos^2 x = 1$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \boxed{\sec^2 x}$$

Ex.4 Differentiate $y = \sec x$.

Recall: $y = \sec x = \frac{1}{\cos x}$

$f(x) = 1 \qquad g(x) = \cos x$

$f'(x) = 0 \qquad g'(x) = -\sin x$

$$y' = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}$$

$$= \frac{0(\cos x) - 1(-\sin x)}{(\cos x)^2}$$

$$= \frac{\sin x}{\cos^2 x} = \frac{1}{\cos x} \cdot \frac{\sin x}{\cos x} = \boxed{\sec x \tan x}$$

Ex.5 Differentiate $y = \cot x$.

Recall: $y = \cot x = \frac{\cos x}{\sin x}$

$f(x) = \cos x \qquad g(x) = \sin x$

$f'(x) = -\sin x \qquad g'(x) = \cos x$

$$y' = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}$$

$$= \frac{(-\sin x)(\sin x) - (\cos x)(\cos x)}{(\sin x)^2}$$

$$= \frac{-\sin^2 x - \cos^2 x}{\sin^2 x} = \frac{-(\sin^2 x + \cos^2 x)}{\sin^2 x} = \frac{-1}{\sin^2 x} = \boxed{-\csc^2 x}$$

Ex. 6 Differentiate $y = \csc x$.

<u>Recall</u>: $y = \csc x = \frac{1}{\sin x}$

$f(x) = 1 \qquad g(x) = \sin x$

$f'(x) = 0 \qquad g'(x) = \cos x$

$$y' = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2} = \frac{0\cdot(\sin x) - 1\cdot(\cos x)}{\sin^2 x} = \frac{-\cos}{\sin^2 x}$$

$$= -\frac{1}{\sin x}\cdot\frac{\cos x}{\sin x}$$

$$= \boxed{-\csc x \cot x}$$

$*$ <u>Derivatives of Trig Functions</u>:

The "co"'s are the jerks and derivative changes sign.

$(\sin x)' = \cos x \qquad (\cos x)' = -\sin x$

$(\tan x)' = \sec^2 x \qquad (\cot x)' = -\csc^2 x$

$(\sec x)' = \sec x \tan x \qquad (\csc x)' = -\csc x \cot x$

Ex. 7 Find the equation of the tangent line to the graph of $y = x^2 \sec x$ at $x = \frac{\pi}{3}$.

- First, find y'.

<u>M1</u>: Product Rule

$f(x) = x^2 \qquad g(x) = \sec x$

$f'(x) = 2x \qquad g'(x) = \sec x \tan x$

$$y' = f'(x)g(x) + f(x)g'(x)$$
$$= 2x\sec x + x^2 \sec x \tan x$$

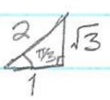
<u>M2</u>: Quotient Rule $\left(y = \frac{x^2}{\cos x}\right)$

$f(x) = x^2 \qquad g(x) = \cos x$

$f'(x) = 2x \qquad g'(x) = -\sin x$

$$y' = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}$$
$$= \frac{2x\cos x + x^2 \sin x}{\cos^2 x}$$
$$= 2x\sec x + x^2 \sec x \tan x$$

- Evaluate y' at $x = \frac{\pi}{3}$.



$\sec\left(\frac{\pi}{3}\right) = \frac{2}{1} = 2$

$\tan\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{1} = \sqrt{3}$

$$y'\Big|_{\frac{\pi}{3}} = 2\left(\frac{\pi}{3}\right)\sec\left(\frac{\pi}{3}\right) + \left(\frac{\pi}{3}\right)^2 \sec\left(\frac{\pi}{3}\right)\tan\left(\frac{\pi}{3}\right)$$
$$= 2\left(\frac{\pi}{3}\right)(2) + \frac{\pi^2}{9}(2)(\sqrt{3})$$
$$= \frac{4}{3}\pi + \frac{2\sqrt{3}}{9}\pi^2$$

- Use point $\left(\frac{\pi}{3}, y\big|_{x=\frac{\pi}{3}}\right)$ to find y-intercept.

$$y\big|_{x=\frac{\pi}{3}} = \left(\frac{\pi}{3}\right)^2 \sec\left(\frac{\pi}{3}\right) = \frac{2}{9}\pi^2$$

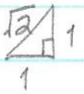
$$y = mx + b$$

$$\frac{2}{9}\pi^2 = \left(\frac{4}{9}\pi + \frac{2\sqrt{3}}{9}\pi^2\right)\left(\frac{\pi}{3}\right) + b$$

$$b = \frac{2}{9}\pi^2 - \left(\frac{4}{27}\pi^2 + \frac{2\sqrt{3}}{27}\pi^3\right) = \frac{2}{27}\pi^2 - \frac{2\sqrt{3}}{27}\pi^3$$

$$\boxed{y = \left(\frac{4}{3}\pi + \frac{2\sqrt{3}}{9}\pi^2\right)x + \frac{2}{27}\pi^2 - \frac{2\sqrt{3}}{27}\pi^3}$$

Ex.8 Differentiate $f(x) = \frac{x\cos x + 1}{\cos x}$. Evaluate at $x = \frac{5\pi}{4}$.



$$f(x) = \frac{x\cos x + 1}{\cos x} = \frac{x\cos x}{\cos x} + \frac{1}{\cos x} = x + \sec x$$

$$f'(x) = 1 + \sec x \tan x$$

$$f'\left(\tfrac{5\pi}{4}\right) = 1 + \sec\left(\tfrac{5\pi}{4}\right)\tan\left(\tfrac{5\pi}{4}\right)$$

$$= 1 + (-\sqrt{2})(1)$$

$$= \boxed{1-\sqrt{2}}$$

Ex.9 Differentiate $f(\theta) = \frac{\sin\theta + \cos\theta}{\theta}$.

$g(\theta) = \sin\theta + \cos\theta \qquad h(\theta) = \theta$

$g'(\theta) = \cos\theta - \sin\theta \qquad h'(\theta) = 1$

$$f'(\theta) = \frac{g'(\theta)h(\theta) - g(\theta)h'(\theta)}{(h(\theta))^2}$$

$$= \frac{(\cos\theta - \sin\theta)(\theta) - (\sin\theta + \cos\theta)(1)}{\theta^2}$$

$$= \frac{\theta\cos\theta - \theta\sin\theta - \sin\theta - \cos\theta}{\theta^2}$$

Ex.10 Differentiate $f(y) = \frac{e^y}{y - e^y}$. Evaluate at $y = 0$.

$g(y) = e^y \qquad h(y) = y - e^y$

$g'(y) = e^y \qquad h'(y) = 1 - e^y$

$$f'(y) = \frac{g'(y)h(y) - g(y)h'(y)}{(h(y))^2}$$

$$= \frac{e^y(y - e^y) - e^y(1 - e^y)}{(y - e^y)^2}$$

$$= \frac{e^y(y - e^y - 1 + e^y)}{(y - e^y)^2}$$

$$= \frac{e^y(y-1)}{(y - e^y)^2}$$

$$f'(0) = \frac{e^0(0-1)}{(0 - e^0)^2} = \frac{(1)(-1)}{(0-1)^2} = \frac{-1}{(-1)^2} = \boxed{-1}$$

* Quiz on Friday at beginning of class over quotient and product rules.