Lesson 11: The Chain Rule

Want to compute the derivative of composite functions.

Recall: $(f\circ g)(x) = f(g(x))$, $(f\circ f)(x) = f(f(x))$, etc.

Ex.1 If $f(x) = \frac{1}{x}$ and $g(x) = \frac{x+1}{x-1}$, find

(a) $(f\circ g)(x) = f(g(x)) = f\left(\frac{x+1}{x-1}\right) = \frac{1}{\left(\frac{x+1}{x-1}\right)} = \boxed{\frac{x-1}{x+1}}$

(b) $(g\circ f)(x) = g(f(x)) = g\left(\frac{1}{x}\right) = \frac{\frac{1}{x}+1}{\frac{1}{x}-1} = \frac{\frac{1+x}{x}}{\frac{1-x}{x}} = \frac{1+x}{x}\cdot\frac{x}{1-x} = \boxed{\frac{1+x}{1-x}}$

(c) $(f\circ f)(x) = f(f(x)) = f\left(\frac{1}{x}\right) = \frac{1}{\left(\frac{1}{x}\right)} = 1\cdot\frac{x}{1} = \boxed{x}$

(d) $(g\circ g)(x) = g(g(x)) = g\left(\frac{x+1}{x-1}\right)$

$$= \frac{\frac{x+1}{x-1}+1}{\frac{x+1}{x-1}-1} = \frac{\frac{x+1+(x-1)}{x-1}}{\frac{x+1-(x-1)}{x-1}} = \frac{\frac{2x}{x-1}}{\frac{2}{x-1}}$$

$$= \frac{2x}{x-1}\cdot\frac{x-1}{2} = \frac{2x}{2} = \boxed{x}$$

Ex.2 If $f(x) = \sin(x)$ and $g(x) = x^2$, find

(a) $(f\circ g)(x) = f(g(x)) = f(x^2) = \boxed{\sin(x^2)}$

(b) $(g\circ f)(x) = g(f(x)) = g(\sin(x)) = (\sin(x))^2 = \boxed{\sin^2(x)}$

(c) $(f\circ f)(x) = f(f(x)) = f(\sin(x)) = \boxed{\sin(\sin(x))}$

(d) $(g\circ g)(x) = g(g(x)) = g(x^2) = (x^2)^2 = x^{2\cdot 2} = \boxed{x^4}$

Note: $\sin(x^2)$; $\sin^2(x) = \sin(x)\cdot\sin(x)$; and $\sin(\sin(x))$ are all different!

"Derivative of the outside of the inside times the derivative of the inside."

* **Chain Rule:** $\frac{d}{dx}\left[f(g(x))\right] = f'(g(x))\cdot g'(x)$

OR: If $y = f(x)$ is a function of x, and $x = g(t)$ is a function of t, then $y = f(x) = f(g(t))$ is a composite function of t, and

$$\frac{dy}{dt} = \frac{dy}{dx}\cdot\frac{dx}{dt}$$

↳ This form will be incredibly relevant for lessons 15 and 16 on related rates.

Ex. 3 Differentiate $y = (x^2 + x)^3$.

- Identify your inside and outside functions.

  Inside: $g(x) = x^2 + x$ Outside: $f(x) = x^3$

  $g'(x) = 2x + 1$ $f'(x) = 3x^2$

- Remember that x is a "dummy" variable. It does not matter that g and f are written in terms of x. We could also write them in terms of t. All that matters is that we can write

  $$y = f(g(x)) = f(x^2 + x) = (x^2 + x)^3. \checkmark$$

- Now we can use the chain rule.

  $$\begin{aligned} y' &= f'(g(x)) \cdot g'(x) \\ &= f'(x^2 + x) \cdot (2x + 1) \\ &= \boxed{3 \cdot (x^2 + x)^2 \cdot (2x + 1)} \end{aligned}$$

Ex. 4 Differentiate $y = \sqrt{1 - x + x^3}$.

- Inside: $g(x) = 1 - x + x^3$ Outside: $f(x) = \sqrt{x} = x^{1/2}$

  $g'(x) = -1 + 3x^2$

  $$\begin{aligned} f'(x) &= \tfrac{1}{2} x^{\frac{1}{2} - 1} \\ &= \tfrac{1}{2} x^{-1/2} = \frac{1}{2x^{1/2}} \\ &= \frac{1}{2\sqrt{x}} \end{aligned}$$

- $y = f(g(x)) = f(1 - x + x^3) = \sqrt{1 - x + x^3}$ ✓

- $$\begin{aligned} y' &= f'(g(x)) \cdot g'(x) \\ &= f'(1 - x + x^3) \cdot (-1 + 3x^2) \\ &= \frac{1}{2\sqrt{1 - x + x^3}} (-1 + 3x^2) \\ &= \boxed{\frac{3x^2 - 1}{2\sqrt{1 - x + x^3}}} \end{aligned}$$

Ex.5 Differentiate $f(t) = \frac{1}{e^t}$ using chain rule.

- Rewrite $f(t) = \frac{1}{e^t} = e^{-t}$.
- Inside: $g(t) = -t$ Outside: $h(t) = e^t$
  $g'(t) = -1$ $h'(t) = e^t$
- $f(t) = h(g(t)) = h(-t) = e^{-t} = \frac{1}{e^t}$ ✓
- $f'(t) = h'(g(t)) \cdot g'(t)$
  $= h'(-t) \cdot (-1) = -h'(-t)$
  $= \boxed{-e^{-t}} = -\frac{1}{e^t}$ (Can check with Quotient Rule)

Ex.6 Differentiate $y = \sin(x^2)$.

- Inside: $g(x) = x^2$ Outside: $f(x) = \sin(x)$
  $g'(x) = 2x$ $f'(x) = \cos(x)$
- $y = f(g(x)) = f(x^2) = \sin(x^2)$ ✓
- $y' = f'(g(x)) \cdot g'(x)$
  $= f'(x^2) \cdot (2x) = \cos(x^2) \cdot 2x = \boxed{2x \cos(x^2)}$

Ex.7 Differentiate $y = \cos^2(x)$.

- Rewrite to make it clearer: $y = \cos^2(x) = (\cos(x))^2$.
- Inside: $g(x) = \cos(x)$ Outside: $f(x) = x^2$
  $g'(x) = -\sin(x)$ $f'(x) = 2x$
- $y = f(g(x)) = f(\cos(x)) = (\cos(x))^2 = \cos^2(x)$ ✓
- $y' = f'(g(x)) \cdot g'(x)$
  $= f'(\cos(x)) \cdot (-\sin(x)) = -\sin(x) \cdot f'(\cos(x))$
  $= -\sin(x) \cdot (2\cos(x)) = \boxed{-2\sin(x)\cos(x)}$

Ex.8 Differentiate $y = \left(e^x + \frac{1}{x}\right)^{-2/3}$.

- Inside: $g(x) = e^x + \frac{1}{x} = e^x + x^{-1}$ Outside: $f(x) = x^{-2/3}$
  $g'(x) = e^x + (-1)x^{-1-1}$ $f'(x) = \left(-\frac{2}{3}\right)x^{-\frac{2}{3}-1}$
  $= e^x - x^{-2}$ $= \left(-\frac{2}{3}\right)x^{-5/3}$
  $= e^x - \frac{1}{x^2}$

- $y = f(g(x)) = f(e^x + \frac{1}{x}) = (e^x + \frac{1}{x})^{-2/3}$
- $y' = f'(g(x)) \cdot g'(x)$
  $= f'(e^x + \frac{1}{x}) \cdot (e^x - \frac{1}{x^2})$
  $= (-\frac{2}{3})(e^x + \frac{1}{x})^{-5/3}(e^x - \frac{1}{x^2}) = \boxed{-\frac{2}{3}(e^x - \frac{1}{x^2})(e^x + \frac{1}{x})^{-5/3}}$

Ex. 9 Differentiate $y = \tan(\tan(x))$.

- Inside: $g(x) = \tan(x)$ Outside: $f(x) = \tan(x)$
  $g'(x) = \sec^2(x)$ $f'(x) = \sec^2(x)$
- $y = f(g(x)) = f(\tan(x)) = \tan(\tan(x))$ ✓
- $y' = f'(g(x)) \cdot g'(x)$
  $= f'(\tan(x)) \cdot (\sec^2(x))$
  $= \sec^2(\tan(x)) \cdot \sec^2(x)$

(Note: $y' = (\sec(\tan(x)))^2 \cdot (\sec(x))^2$
$= (\sec(\tan(x)))(\sec(\tan(x))) \cdot \sec(x) \cdot \sec(x)$)

Ex. 10 Find the equation of the tangent line to the graph of $y = 2(\sin(x) + \sqrt[3]{x^5})^2$ at $x = 0$.

- Need to find the slope of the tan line at $x = 0$, so take derivative and evaluate at $x=0$.
  - Inside: $g(x) = (\sin(x) + \sqrt[3]{x^5})$ Outside: $f(x) = 2x^2$
    $= \sin(x) + x^{5/3}$ $f'(x) = 4x$
    $g'(x) = \cos(x) + \frac{5}{3}x^{2/3}$
  - $y = f(g(x)) = f(\sin(x) + \sqrt[3]{x^5}) = 2(\sin(x) + \sqrt[3]{x^5})^2$ ✓
  - $y' = f'(g(x)) \cdot g'(x)$
    $= f'(\sin(x) + \sqrt[3]{x^5}) \cdot (\cos(x) + \frac{5}{3}x^{2/3})$
    $= 4(\sin(x) + \sqrt[3]{x^5})(\cos(x) + \frac{5}{3}x^{2/3})$
  - @ $x = 0$: $y' = 4(\sin(0) + \sqrt[3]{0^5})(\cos(0) + \frac{5}{3}(0)^{2/3}) = 0$
- Need point on tan line: $(0, y|_{x=0}) = (0, 0)$
  ↳ $2(\sin(0) + \sqrt[3]{0^5})^2 = 0$
- $y = mx + b \Rightarrow 0 = 0 \cdot 0 + b \Rightarrow b = 0$ $\boxed{y = 0}$