Lesson 12: More Chain Rule; Derivative of ln(x)

* $\frac{d}{dx}(\ln(x)) = \frac{1}{x}$ (The derivative for $\log_b(x)$ for any other base has a different derivative.)

** Because ln(x) is a function, there will almost always be a chain rule involved, but we can often use log properties to avoid complicated chain rules. **

Recall: Log Properties – $\ln(xy) = \ln(x) + \ln(y)$ → No more product rule.
$\ln\left(\frac{x}{y}\right) = \ln(x) - \ln(y)$ → No more quotient rule.
$\ln(x^y) = y\ln(x)$ → One fewer chain rule.

Ex.1 Differentiate $y = \ln(xe^x)$.

Method 1: Differentiate as written.

Chain Rule: $Out = \ln(x)$
$Out' = \frac{1}{x}$

$In = \underbrace{xe^x}_{\text{product rule}}$

$$In' = \frac{d}{dx}(x)\cdot e^x + x\cdot\frac{d}{dx}(e^x)$$
$$= 1\cdot x^{\cancel{0}\,1}\cdot e^x + xe^x$$
$$= e^x + xe^x$$

$$y' = Out'(In)\cdot In'$$
$$= Out'(xe^x)\cdot(e^x + xe^x)$$
$$= \frac{1}{xe^x}(e^x + xe^x)$$
$$= \frac{e^x(1+x)}{xe^x}$$
$$= \frac{1+x}{x}$$
$$= \frac{1}{x} + \frac{x}{x}$$
$$= \boxed{\frac{1}{x} + 1}$$

Method 2: Use log rules first.

$$y = \ln(xe^x)$$
$$= \ln(x) + \ln(e^x)$$
$$= \ln(x) + x\cdot\ln(e)$$
$$= \ln(x) + x\cdot(1)$$
$$= \ln(x) + x$$

Then differentiate.

$$y' = \frac{d}{dx}(\ln(x)) + \frac{d}{dx}(x)$$
$$= \boxed{\frac{1}{x} + 1}$$

Ex.2 Find the derivative of $y = x\sqrt{(1-x^2)^3}$ at $x = 0$.

HW#2

- We can write $y = \underbrace{x}\,\underbrace{(1-x^2)^{3/2}}$.

The two parts cannot be combined, so the first thing we have to do is the product rule.

- Product Rule: $y' = \frac{d}{dx}(x)\cdot(1-x^2)^{3/2} + x\cdot\underbrace{\frac{d}{dx}\left((1-x^2)^{3/2}\right)}_{\text{chain rule}}$

- Chain Rule: $\frac{d}{dx}\left(\underbrace{(1-x^2)}^{3/2}\right)$ $Out = x^{3/2}$ $Out' = \frac{3}{2}x^{1/2}$ $In = 1-x^2$ $In' = 0-2x = -2x$

$1-x^2$ is trapped in parentheses

$$= Out'(In)\cdot In'$$
$$= Out'(1-x^2)\cdot(-2x)$$
$$= \frac{3}{\cancel{2}}(1-x^2)^{1/2}(-\cancel{2}x)$$
$$= -3x(1-x^2)^{1/2}$$

- $y' = 1\cdot(1-x^2)^{3/2} + x\left(-3x(1-x^2)^{1/2}\right)$
$= (1-x^2)^{3/2} - 3x^2(1-x^2)^{1/2}$

- We could simplify more, but since we're evaluating, we can leave it this way.

- $y'|_{x=0} = (1-0^2)^{3/2} - 3(0)^2(1-0^2)^{1/2}$
$= (1-0)^{3/2} - 3(0)(1-0)^{1/2}$
$= 1^{3/2} - 3(0)(1)^{1/2}$
$= 1 - 3(0)(1)$
$= 1 - 0$
$= \boxed{1}$

Ex.3 Find the derivative of $y = \underbrace{e^{-x^2}}\,\underbrace{\sin(3x)}$.

HW#6

- We have two pieces here being multiplied together, so the "biggest" thing happening in the function is a product.

- Product Rule: $y' = \frac{d}{dx}(e^{-x^2})\cdot\sin(3x) + e^{-x^2}\cdot\frac{d}{dx}(\sin(3x))$

The inside is more than just x, so we have a chain rule.

- Chain Rule: $\frac{d}{dx}(e^{-x^2})$: $Out = e^{x}$ $Out' = e^{x}$ $In = -x^2$ $In' = -2x$ → Check: $Out(In) = Out(-x^2) = e^{-x^2}$ ✓

$$= Out'(In)\cdot In'$$
$$= Out'(-x^2)\cdot(-2x)$$
$$= e^{-x^2}(-2x)$$
$$= -2xe^{-x^2}$$

- Chain Rule: $\frac{d}{dx}(\sin(3x))$: Out $= \sin(x)$ In $= 3x$
  Out$' = \cos(x)$ $\cdot$ In$' = 3$

  $\hookrightarrow = \text{Out}'(\text{In}) \cdot \text{In}'$
  $= \text{Out}'(3x) \cdot (3)$
  $= 3\cos(3x)$

- Back to Product Rule:

$$y' = (-2xe^{-x^2}) \cdot \sin(3x) + e^{-x^2} \cdot (3\cos(3x))$$
$$= -2xe^{-x^2}\sin(3x) + 3e^{-x^2}\cos(3x)$$
$$= \boxed{e^{-x^2}(2x\sin(3x) + 3\cos(3x))}$$

Note: ① When the exponent on e is anything more complicated than just x, we have a chain rule. It always looks like $\boxed{\frac{d}{dx}(e^{f(x)}) = f'(x) \cdot e^{f(x)}}$.

The exponent on e never changes.

** This is my favorite type of derivative. **

② The chain rule with $\sin(2x)$, $\cos(3x)$, etc. is the most commonly overlooked chain rule. Always look for sneaky chain rules!

Ex. 4 Find the derivative of $y = (2x-1)^3 \cdot (x^3+2)^2$ at $x = 1$.

HW#1

- There are two pieces that are easily identified by parentheses and exponents here being multiplied together, so we'll start with the product rule and then have chain rules.

- Product Rule: $y' = \underbrace{\frac{d}{dx}((2x-1)^3)}_{\text{chain}} \cdot (x^3+2)^2 + (2x-1)^3 \cdot \underbrace{\frac{d}{dx}((x^3+2)^2)}_{\text{chain}}$

  - Chain: $\frac{d}{dx}((2x-1)^3)$: Out $= x^3$ In $= 2x-1$
    Out$' = 3x^2$ In$' = 2$

    $\hookrightarrow = \text{Out}'(\text{In}) \cdot \text{In}'$
    $= \text{Out}'(2x-1) \cdot 2$
    $= 3(2x-1)^2 \cdot 2$
    $= 6(2x-1)^2$

  - Chain: $\frac{d}{dx}((x^3+2)^2)$: Out $= x^2$ In $= x^3+2$
    Out$' = 2x$ $\cdot$ In$' = 3x^2$

    $\hookrightarrow = \text{Out}'(\text{In}) \cdot \text{In}'$
    $= \text{Out}'(x^3+2) \cdot 3x^2$
    $= 2(x^3+2) \cdot (3x^2)$
    $= 6x^2(x^3+2)$

- Back to product rule:

$$y' = (6(2x-1)^2)(x^3+2)^2 + (2x-1)^3(6x^2(x^3+2))$$
$$= 6(2x-1)^2(x^3+2)^2 + 6x^2(2x-1)^3(x^3+2)$$

- Evaluate at $x=1$:

$$y' = 6(2(1)-1)^2(1^3+2)^2 + 6(1)^2(2(1)-1)^3(1^3+2)$$
$$= 6(2-1)^2(1+2)^2 + 6(1)(2-1)^3(1+2)$$
$$= 6(1)^2(3)^2 + 6(1)(1)^3(3)$$
$$= 6(1)(9) + 6(1)(1)(3)$$
$$= 54 + 18$$
$$= \boxed{72}$$

Ex.5 Find the derivative of $y = \ln\sqrt{\frac{2x+1}{x^3+1}}$.

HW#13 HW#11

- As written, we would start by doing a chain rule with $Out = \ln(x)$ and $In = \sqrt{\frac{2x+1}{x^3+1}}$. Then we have to do another chain rule to find the derivative of In with $out = \sqrt{x}$ and $in = \frac{2x+1}{x^3+1}$. THEN, for the derivative of in, we have a quotient rule with $Top = 2x+1$ and $Bottom = x^3+1$.
  That sounds really horrible, so we'll use log rules to rewrite y before doing the derivative.

- $y = \ln\sqrt{\frac{2x+1}{x^3+1}}$
  $= \ln\left(\frac{2x+1}{x^3+1}\right)^{1/2}$
  $= \frac{1}{2}\cdot\ln\left(\frac{2x+1}{x^3+1}\right)$ → Now, we have a chain rule with $Out = \frac{1}{2}\ln(x)$ and $In = \frac{2x+1}{x^3+1}$, so the derivative of In still needs a quotient rule. Let's rewrite more.
  $= \frac{1}{2}(\ln(2x+1) - \ln(x^3+1))$
  $= \frac{1}{2}\ln(2x+1) - \frac{1}{2}\ln(x^3+1)$ → We can't rewrite any further, so we still have to do 2 chain rules, but they are not nested, and we don't have a quotient rule.

- $y' = \underbrace{\frac{d}{dx}\left(\frac{1}{2}\ln(2x+1)\right)}_{\text{chain}} - \underbrace{\frac{d}{dx}\left(\frac{1}{2}\ln(x^3+1)\right)}_{\text{chain}}$ (<u>OR</u> $= \frac{1}{2}\left[\frac{d}{dx}(\ln(2x+1)) - \frac{d}{dx}(\ln(x^3+1))\right]$
  <u>OR</u> $= \frac{1}{2}\cdot\frac{d}{dx}(\ln(2x+1)) - \frac{1}{2}\frac{d}{dx}(\ln(x^3+1))$)

– Chain: $\frac{d}{dx}\left(\frac{1}{2}\ln(2x+1)\right)$: $\quad Out = \frac{1}{2}\ln(x) \quad In = 2x+1$

$Out' = \frac{1}{2}\cdot\frac{1}{x} = \frac{1}{2x} \qquad In' = 2$

$$= Out'(In)\cdot In'$$
$$= Out'(2x+1)\cdot 2$$
$$= \frac{1}{\cancel{2}(2x+1)}\cdot\cancel{2}$$
$$= \frac{1}{2x+1}$$

– Chain: $\frac{d}{dx}\left(\frac{1}{2}\ln(x^3+1)\right)$: $\quad Out = \frac{1}{2}\ln(x) \quad In = x^3+1$

$Out' = \frac{1}{2}\cdot\frac{1}{x} = \frac{1}{2x} \qquad In' = 3x^2$

$$= Out'(In)\cdot In'$$
$$= Out'(x^3+1)\cdot(3x^2)$$
$$= \frac{1}{2(x^3+1)}\,(3x^2)$$
$$= \frac{3x^2}{2(x^3+1)}$$

- $\boxed{y' = \frac{1}{2x+1} - \frac{3x^2}{2(x^3+1)}}$

Ex.6 Find the derivative of $y = \frac{\sqrt{1-x^2}}{\ln(x)}$ at $x = e^{-2}$.

- Here, the "biggest" thing happening is the division of 2 functions, so we start with a quotient rule.

HW#3 HW#12

- Quotient Rule: $Top = \underbrace{(1-x^2)^{1/2}}_{\text{chain rule}} \qquad Bottom = \ln(x)$

$Bottom' = \frac{1}{x}$

– Chain: $\frac{d}{dx}\left((1-x^2)^{1/2}\right)$: $\quad Out = x^{1/2} \quad In = 1-x^2$

$Out' = \frac{1}{2}x^{-1/2} \qquad In' = -2x$

$$= Out'(In)\cdot In'$$
$$= Out'(1-x^2)\cdot(-2x)$$
$$= \frac{1}{\cancel{2}}(1-x^2)^{-1/2}(-\cancel{2}x)$$
$$= -\frac{x}{\sqrt{1-x^2}}$$

- Back to Quotient: $\quad Top = (1-x^2)^{1/2} \qquad Bottom = \ln(x)$

$Top' = -\frac{x}{\sqrt{1-x^2}} \qquad Bottom' = \frac{1}{x}$

$$y' = \frac{Top'\cdot Bottom - Top\cdot Bottom'}{(Bottom)^2}$$
$$= \frac{\left(-\frac{x}{\sqrt{1-x^2}}\right)(\ln(x)) - (1-x^2)^{1/2}\cdot\frac{1}{x}}{(\ln(x))^2}$$
$$= \frac{\frac{-x\ln(x)}{\sqrt{1-x^2}} - \frac{\sqrt{1-x^2}}{x}}{(\ln(x))^2}$$

- Evaluate at $x=e^{-2}$:

$$y'|_{x=e^{-2}} = \frac{\frac{-e^{-2}\cdot\ln(e^{-2})}{\sqrt{1-(e^{-2})^2}} - \frac{\sqrt{1-(e^{-2})^2}}{e^{-2}}}{(\ln(e^{-2}))^2}$$

Note: $\ln(e^{-2}) = -2\cdot\ln(e)$
$= -2$

$$= \frac{\frac{-e^{-2}(-2)}{\sqrt{1-e^{-4}}} - \frac{\sqrt{1-e^{-4}}}{e^{-2}}}{(-2)^2}$$

$$= \frac{\frac{2e^{-2}}{\sqrt{1-e^{-4}}} - \frac{\sqrt{1-e^{-4}}}{e^{-2}}}{4}$$

$$= \frac{1}{4}\left[\frac{2e^{-4}}{\sqrt{1-e^{-4}}} - \frac{\sqrt{1-e^{-4}}}{e^{-2}}\right]$$

$$= \frac{2e^{-4}}{4\sqrt{1-e^{-4}}} - \frac{\sqrt{1-e^{-4}}}{4e^{-2}}$$

$$= \frac{e^{-4}}{2\sqrt{1-e^{-4}}} - \frac{\sqrt{1-e^{-4}}}{4e^{-2}}$$

$$= \boxed{\frac{1}{2e^4\sqrt{1-e^{-4}}} - \frac{1}{4}e^2\sqrt{1-e^{-4}}}$$

Ex.7 Differentiate $y = 2\cot(3x^2+1)$.

- Straight up chain rule here: Out $= 2\cot(x)$ In $= 3x^2+1$
Out$' = -2\csc^2(x)$ In$' = 6x$

HW #4

- $y' = \text{Out}'(\text{In})\cdot\text{In}'$
$= \text{Out}'(3x^2+1)\cdot(6x)$
$= -2\csc^2(3x^2+1)\cdot 6x$
$= \boxed{-12x\csc^2(3x^2+1)}$

In LC: -12 * x * (csc(3 * x^2 + 1))^2

Ex.8 Find the derivative of $y = 3\sin^2(2x)$ at $x = \frac{\pi}{3}$.

HW #5

- Let's rewrite to make it clearer what the order of operations is: $y = 3(\sin(2x))^2$.
The "biggest" thing happening is a chain rule.

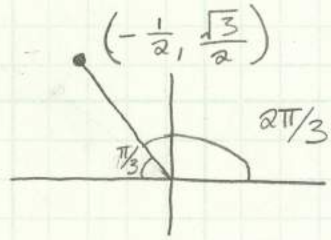
- Chain: Out $= 3x^2$ In $= \sin(2x)$
Out$' = 6x$

Another chain rule!
out $= \sin(x)$ in $= 2x$
out$' = \cos(x)$ in$' = 2$

In$' = $ out$'$(in) $\cdot$ in$'$
$=$ out$'(2x)\cdot 2$
$= \cos(2x)\cdot 2 = 2\cos(2x)$

- $y' = Out'(In) \cdot In'$
  $= Out'(\sin(2x)) \cdot (2\cos(2x))$
  $= 6\sin(2x) \cdot (2\cos(2x))$
  $= 12\sin(2x)\cos(2x)$

- Evaluate at $x = \frac{\pi}{3}$:



$$
\begin{aligned}
y'\big|_{x=\frac{\pi}{3}} &= 12\sin\left(2\cdot\tfrac{\pi}{3}\right)\cos\left(2\cdot\tfrac{\pi}{3}\right) \\
&= 12\sin\left(\tfrac{2\pi}{3}\right)\cos\left(\tfrac{2\pi}{3}\right) \\
&= 12\left(\tfrac{\sqrt{3}}{2}\right)\left(-\tfrac{1}{2}\right) \\
&= \frac{-12\sqrt{3}}{4} \\
&= \boxed{-3\sqrt{3}}
\end{aligned}
$$

Ex. 9 Find the equation for the tangent line to $y = \underline{(x^2+1)}\,\underline{\sqrt{4x}}$ at $x = 1$.

HW#7

① Find the slope for the tangent line: $m = y'\big|_{x=1}$.

- To find y', start with the Product Rule:

$$y' = \frac{d}{dx}(x^2+1)\cdot\sqrt{4x} + (x^2+1)\cdot\underbrace{\frac{d}{dx}\left(\sqrt{4x}\right)}_{\text{chain}}$$

- Chain: $\frac{d}{dx}\left(\sqrt{4x}\right)$: $Out = x^{1/2}$, $In = 4x$; $Out' = \frac{1}{2}x^{-1/2}$, $In' = 4$

$$
\begin{aligned}
&= Out'(In)\cdot In' \\
&= Out'(4x)\cdot 4 \\
&= \tfrac{1}{\cancel{2}}(4x)^{-1/2}\cdot\cancel{4} \\
&= 2(4x)^{-1/2} \\
&= \frac{2}{\sqrt{4x}}
\end{aligned}
$$

- $y' = (2x)\cdot\sqrt{4x} + (x^2+1)\cdot\frac{2}{\sqrt{4x}}$

- Evaluate at $x = 1$:

$$
\begin{aligned}
y' &= (2\cdot 1)\sqrt{4\cdot 1} + (1^2+1)\cdot\frac{2}{\sqrt{4\cdot 1}} \\
&= 2\sqrt{4} + (2)\cdot\frac{2}{\sqrt{4}} \\
&= 2(2) + \cancel{2}\cdot\frac{2}{\cancel{2}} \\
&= 4 + 2 \\
&= 6
\end{aligned}
$$

- Slope $m = y'\big|_{x=1} = 6$

② Find the y-coordinate at $x = 1$:

$$y\big|_{x=1} = (1^2+1)\sqrt{4\cdot 1} = (1+1)\sqrt{4} = 2(2) = 4$$

(3) Use the slope $m=6$ and point $(1,4)$ to find the equation for the tangent line:

$$y - y_1 = m(x - x_1)$$
$$y - 4 = 6(x-1)$$
$$y - 4 = 6x - 6$$
$$y = 6x - 6 + 4$$
$$\boxed{y = 6x - 2}$$

Ex.10 Differentiate $y = \frac{2x}{(x^2+1)^3}$ and find $y'(1)$.

HW#9

Method 1: Proceed as written, so we start with the quotient rule: Top $= 2x$ Bottom $= \underbrace{(x^2+1)^3}_{\text{Chain}}$

Top' $= 2$

• Chain: $\frac{d}{dx}((x^2+1)^3)$: Out $= x^3$ In $= x^2+1$

Out' $= 3x^2$ In' $= 2x$

$$= \text{Out}'(\text{In}) \cdot \text{In}'$$
$$= \text{Out}'(x^2+1) \cdot 2x$$
$$= 3(x^2+1)^2 \cdot 2x$$
$$= 6x(x^2+1)^2$$

• Continue with quotient rule:

Top $= 2x$ Bottom $= (x^2+1)^3$

Top' $= 2$ Bottom' $= 6x(x^2+1)^2$

$$y' = \frac{\text{Top}' \cdot \text{Bottom} - \text{Top} \cdot \text{Bottom}'}{(\text{Bottom})^2}$$
$$= \frac{2(x^2+1)^3 - 2(6x(x^2+1)^2)}{((x^2+1)^3)^2}$$
$$= \frac{2(x^2+1)^3 - 12x(x^2+1)^2}{(x^2+1)^6}$$

• Evaluate at $x=1$:

$$y'(1) = \frac{2(1^2+1)^3 - 12(1)(1^2+1)^2}{(1^2+1)^6}$$
$$= \frac{2(2^3) - 12(2^2)}{2^6}$$
$$= \frac{16 - 48}{64}$$
$$= \frac{-32}{64}$$
$$= \boxed{-\frac{1}{2}}$$

Method 2: Change the quotient into a product by rewriting: $y = \frac{2x}{(x^2+1)^3} = 2x(x^2+1)^{-3}$

• Product Rule: $y' = \frac{d}{dx}(2x)\cdot(x^2+1)^{-3} + 2x \cdot \underbrace{\frac{d}{dx}\left((x^2+1)^{-3}\right)}_{\text{chain}}$

– Chain: $\frac{d}{dx}\left((x^2+1)^{-3}\right)$: $Out = x^{-3}$ $In = x^2+1$
$Out' = -3x^{-4}$ $In' = 2x$

$$\begin{aligned} &\hookrightarrow = Out'(In)\cdot In' \\ &= Out'(x^2+1)\cdot 2x \\ &= -3(x^2+1)^{-4}\cdot 2x \\ &= -6x(x^2+1)^{-4} \end{aligned}$$

• Back to Product Rule:

$$\begin{aligned} y' &= 2(x^2+1)^{-3} + 2x\left(-6x(x^2+1)^{-4}\right) \\ &= \frac{2}{(x^2+1)^3} - \frac{12x^2}{(x^2+1)^4} \end{aligned}$$

• Evaluate at $x=1$:

$$\begin{aligned} y'(1) &= \frac{2}{(1^2+1)^3} - \frac{12(1)^2}{((1)^2+1)^4} \\ &= \frac{2}{2^3} - \frac{12}{2^4} \\ &= \frac{1}{4} - \frac{12}{16} \\ &= \frac{1}{4} - \frac{3}{4} \\ &= \boxed{-\frac{1}{2}} \end{aligned}$$

Ex. 11 Differentiate $y = \sqrt{5x}\,\ln(2x)$.

HW#10

• Product Rule: $y' = \underbrace{\frac{d}{dx}(\sqrt{5x})}_{\text{chain}}\cdot\ln(2x) + \sqrt{5x}\cdot\underbrace{\frac{d}{dx}(\ln(2x))}_{\text{chain}}$

– Chain: $\frac{d}{dx}(\sqrt{5x})$: $Out = x^{1/2}$ $In = 5x$
$Out' = \frac{1}{2}x^{-1/2}$ $In' = 5$

$$\begin{aligned} &\hookrightarrow = Out'(In)\cdot In' \\ &= Out'(5x)\cdot 5 \\ &= \tfrac{1}{2}(5x)^{-1/2}(5) = \frac{5}{2\sqrt{5x}} \end{aligned}$$

– Chain: $\frac{d}{dx}(\ln(2x))$: $Out = \ln(x)$ $In = 2x$
$Out' = \frac{1}{x}$ $In' = 2$

$$\begin{aligned} &\hookrightarrow = Out'(In)\cdot In' \\ &= Out'(2x)\cdot 2 \\ &= \frac{1}{\cancel{2}x}\cdot\cancel{2} = \frac{1}{x} \end{aligned}$$

OR: $\ln(2x) = \ln(2) + \ln(x)$, where $\ln(2)$ is a constant

Then $\frac{d}{dx}(\ln(2x)) = \frac{d}{dx}(\ln(2)) + \frac{d}{dx}(\ln(x))$

$$= 0 + \frac{1}{x}$$
$$= \frac{1}{x}$$

- Back to Product Rule:

$$y' = \frac{5}{2\sqrt{5x}} \cdot \ln(2x) + \sqrt{5x} \cdot \frac{1}{x}$$
$$= \boxed{\frac{5\ln(2x)}{2\sqrt{5x}} + \frac{\sqrt{5x}}{x}}$$

Ex. 12 Differentiate $P = 14e^{-0.004t}$.

HW#8

- The exponent on e is more than just t, so we need a chain rule.
- Chain: $Out = 14e^{t}$ $\quad In = -0.004t$
  $Out' = 14e^{t}$ $\cdot$ $In' = -0.004$

$$P' = Out'(In) \cdot In'$$
$$= Out'(-0.004t)(-0.004)$$
$$= 14e^{-0.004t}(-0.004)$$
$$= 14(-0.004)e^{-0.004t}$$
$$= \boxed{-0.056\,e^{-0.004t}}$$

From now on, the chain rule can happen anywhere, so don't rush when taking derivatives.

Always watch out for sneaky chain rules!