Lesson 13: Higher Order Derivatives

* Notation: $f'(x),\ y',\ \frac{dy}{dx}$ - take derivative one time
$f''(x),\ y'',\ \frac{d^2y}{dx^2}$ - take derivative two times
$f^{(3)}(x),\ y^{(3)},\ \frac{d^3y}{dx^3}$ - take derivative three times
$\vdots$
$f^{(n)}(x),\ y^{(n)},\ \frac{d^ny}{dx^n}$ - take derivative n times

Recall: If $s(t)$ is a position function, $s'(t)=v(t)$ is the velocity function, and $v'(t)=a(t)$ is the acceleration function. Now, we can write $s''(t)=v'(t)=a(t)$.

Ex. 1 Given the position function $s(t)=t^3+4t+5$, find the acceleration when the velocity is 4 m/s.

Velocity: $v(t)=s'(t)=3t^2+4$
Acceleration: $a(t)=v'(t)=s''(t)=6t$

$$v(t)=3t^2+4=4$$
$$3t^2=0$$
$$t=0$$

Velocity is 4 m/s when $t=0$ s. Need to find $a(0)$.

$$a(0)=v'(0)=s''(0)=6(0)=\boxed{0\ m/s^2}$$

Ex. 2 If the second derivative of $f(x)$ is $f''(x)=e^x\cos(x)$, find $f^{(4)}(x)$.

$$f^{(3)}(x)=\frac{d}{dx}(f''(x))=\frac{d}{dx}(e^x\cos(x)) \leftarrow \text{product rule}$$
$$=\cos(x)\frac{d}{dx}(e^x)+e^x\frac{d}{dx}(\cos(x))$$
$$=\cos(x)e^x+e^x(-\sin(x))=e^x(\cos(x)-\sin(x))$$
$$f^{(4)}(x)=\frac{d}{dx}(f^{(3)}(x))=\frac{d}{dx}(e^x(\cos(x)-\sin(x))) \leftarrow \text{product rule}$$
$$=(\cos(x)-\sin(x))\frac{d}{dx}(e^x)+e^x\frac{d}{dx}(\cos(x)-\sin(x))$$
$$=(\cos(x)-\sin(x))e^x+e^x(-\sin(x)-\cos(x))$$
$$=e^x(\cos(x)-\sin(x)-\sin(x)-\cos(x))=\boxed{-2e^x\sin(x)}$$

Ex.3 Given $y = e^{x^2}\ln(\sqrt{x})$, find y''.

- Rewrite: $y = e^{x^2}\ln(x^{\frac{1}{2}}) = \frac{1}{2}e^{x^2}\ln(x)$
- Product Rule: $y' = \frac{1}{2}[\ln(x)\underbrace{\frac{d}{dx}(e^{x^2})}_{\text{chain rule!}} + e^{x^2}\frac{d}{dx}(\ln(x))]$
- Chain Rule: Inside: $g(x) = x^2$, $g'(x) = 2x$ outside: $f(x) = e^x$, $f'(x) = e^x$

$\hookrightarrow \frac{d}{dx}(e^{x^2}) = f'(g(x))\cdot g'(x)$
$= e^{x^2}(2x) = 2xe^{x^2}$ ← power on e does not change!

- $y' = \frac{1}{2}[\ln(x)(2xe^{x^2}) + e^{x^2}\cdot\frac{1}{x}]$
$= \frac{1}{2}[2xe^{x^2}\ln(x) + \frac{1}{x}e^{x^2}]$
$= \frac{1}{2}\underbrace{e^{x^2}}_{\text{one function}}\underbrace{(2x\ln(x) + \frac{1}{x})}_{\text{another function}}$

- Product Rule:
$y'' = \frac{1}{2}[(2x\ln(x) + \frac{1}{x})\overbrace{\frac{d}{dx}(e^{x^2})}^{\text{chain rule from above}} + e^{x^2}\frac{d}{dx}(2x\ln(x) + \frac{1}{x})]$
$= \frac{1}{2}[(2x\ln(x) + \frac{1}{x})(2xe^{x^2}) + e^{x^2}(\underbrace{\frac{d}{dx}(2x\ln(x))}_{\text{product rule}} + \frac{d}{dx}(\frac{1}{x}))]$
$= \frac{1}{2}[(2x\ln(x) + \frac{1}{x})(2xe^{x^2}) + e^{x^2}(2\ln(x) + 2x\cdot\frac{1}{x} + (-x^{-2}))]$
$= \frac{1}{2}e^{x^2}[(2x\ln(x) + \frac{1}{x})(2x) + 2\ln(x) + 2 - \frac{1}{x^2}]$
$= \frac{1}{2}e^{x^2}(4x^2\ln(x) + 2 + 2\ln(x) + 2 - \frac{1}{x^2})$
$= \frac{1}{2}e^{x^2}[(4x^2 + 2)\ln(x) + 4 - \frac{1}{x^2}]$

Ex.4 Given $y = \sqrt{2x} - \ln(x)$, find y''.

- $y' = \underbrace{\frac{d}{dx}(2x)^{1/2}}_{\text{chain rule}} - \frac{d}{dx}(\ln(x))$
$= \frac{1}{2}(2x)^{-1/2}(2) - \frac{1}{x} = (2x)^{-1/2} - x^{-1}$

- $y'' = \underbrace{\frac{d}{dx}(2x)^{-1/2}}_{\text{chain rule}} - \frac{d}{dx}(x^{-1})$
$= -\frac{1}{2}(2x)^{-3/2}(\cancel{2}) - (-x^{-2}) = \boxed{-(2x)^{-3/2} + x^{-2}}$
$= -\frac{1}{(2x)^{3/2}} + \frac{1}{x^2}$

Ex.5 Given $x = y^3 \sin(2y)$, find $\frac{d^2x}{dy^2}$.

- Product rule: $\frac{dx}{dy} = \sin(2y)\frac{d}{dy}(y^3) + y^3 \underbrace{\frac{d}{dy}(\sin(2y))}_{\text{chain rule}}$

$$= \sin(2y)(3y^2) + y^3(\cos(2y)(2))$$
$$= y^2\left(3\sin(2y) + 2y\cos(2y)\right)$$

- Product Rule:

$$\frac{d^2x}{dy^2} = (3\sin(2y) + 2y\cos(2y))\frac{d}{dy}(y^2) + y^2\Big(\underbrace{\frac{d}{dy}(3\sin(2y))}_{\text{chain rule}} + \underbrace{\frac{d}{dy}(2y\cos(2y))}_{\text{product rule}}\Big)$$

$$= (3\sin(2y) + 2y\cos(2y))(2y) + y^2\Big(6\cos(2y) + \underbrace{\frac{d}{dy}(2y\cos(2y))}_{\text{product rule}}\Big)$$

$$\frac{d}{dy}(2y\cos(2y)) = \cos(2y)\frac{d}{dy}(2y) + 2y\frac{d}{dy}(\cos(2y))$$
$$= \cos(2y)(2) + 2y(-\sin(2y)(2))$$
$$= 2\cos(2y) - 4y\sin(2y)$$

$$\frac{d^2x}{dy^2} = 6y\sin(2y) + 4y^2\cos(2y) + 6y^2\cos(2y) + y^2\left(2\cos(2y) - 4y\sin(2y)\right)$$
$$= 6y\sin(2y) + 4y^2\cos(2y) + 6y^2\cos(2y) + 2y^2\cos(2y) - 4y^3\sin(2y)$$
$$= \boxed{12y^2\cos(2y) + (6y - 4y^3)\sin(2y)}$$

Ex.6 Given $\frac{d^3y}{dx^3} = \sec(3x+2)$, find $\frac{d^4y}{dx^4}$.

$$\frac{d^4y}{dx^4} = \frac{d}{dx}\left(\frac{d^3y}{dx^3}\right) = \underbrace{\frac{d}{dx}(\sec(3x+2))}_{\text{chain rule}}$$
$$= 3\sec(3x+2)\tan(3x+2)$$

Ex. 7 Given the velocity $v(t) = \frac{t^2+1}{2t+3}$, $t \geq 0$, find (i) $a(t)$ and (ii) the acceleration when the velocity is 2 m/s.

(i) Quotient Rule:

$$a(t) = v'(t) = \frac{2t(2t+3) - 2(t^2+1)}{(2t+3)^2}$$

$$= \frac{4t^2 + 6t - 2t^2 - 2}{(2t+3)^2}$$

$$= \boxed{\frac{2t^2 + 6t - 2}{(2t+3)^2}}$$

(ii) $v(t) = 2$: $\frac{t^2+1}{2t+3} = 2$

$$t^2 + 1 = 2(2t+3)$$

$$t^2 + 1 = 4t + 6$$

$$t^2 - 4t - 5 = 0$$

$$(t-5)(t+1) = 0$$

$$\Rightarrow \quad t = 5, \quad \cancel{t = -1} \quad (t \geq 0)$$

$$a(5) = \frac{2(5)^2 + 6(5) - 2}{(2(5)+3)^2} = \frac{50+30-2}{(13)^2} = \boxed{\frac{78}{169}} = \boxed{\frac{6}{13}}$$