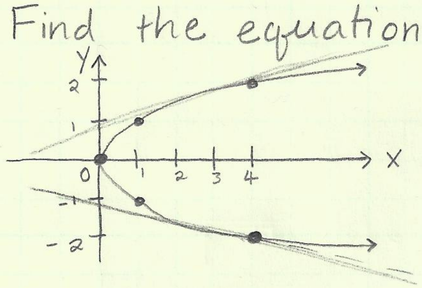
Lesson 14: Implicit Differentiation

* Use when we can't solve for y explicitly.

HW#1
HW#4

Ex.1 Find the equation of the tangent line to $y^2 = x$ at $(4,-2)$.



Note: We always have the equation of a tangent line as $y = mx + b$, where the slope m is $\frac{\text{change in } y}{\text{change in } x}$.

This means the slope of a tangent line is always $\frac{dy}{dx}$.

Looking at the graph, when $x = 4$, $y = -2$ and $y = 2$ satisfy $y^2 = x$, so the x- and y-values are specified to determine which line we're looking for.
This also means that $\frac{dy}{dx}$ can be in terms of x and y.

Now, for the actual derivative, we need to take the derivative with respect to x to find $\frac{dy}{dx}$.

$\frac{d}{dx}(y^2 = x)$

$\frac{d}{dx}(y^2) = \underbrace{\frac{d}{dx}(x)}_{\substack{\text{This is a} \\ \text{normal derivative} \\ \text{like we've done.}}}$

What we have here is basically a chain rule with Out $= y^2$ In $= y$
Out' $= 2y$ In' $= y'$ (or $\frac{dy}{dx}$)

$\frac{d}{dx}(y^2) = 2y \cdot y' = 2y\frac{dy}{dx}$

$$\boxed{2y\frac{dy}{dx} \overset{=1}{=} 1}$$

Now, we solve for $\frac{dy}{dx}$: $\boxed{\frac{dy}{dx} = \frac{1}{2y}}$

① Find the slope for the tangent line at $(\overset{x}{4}, \overset{y}{-2})$:

$$m = \left.\frac{dy}{dx}\right|_{(4,-2)} = \frac{1}{2(-2)} = \boxed{-\frac{1}{4}}$$

② Already know the point $(4,-2)$ is on the line.

③ Use $m = -\frac{1}{4}$ and $(4,-2)$ to find the tangent line:

$$y - y_1 = m(x - x_1)$$
$$y - (-2) = -\tfrac{1}{4}(x - 4)$$
$$y + 2 = -\tfrac{1}{4}x + 1$$
$$\boxed{y = -\tfrac{1}{4}x - 1}$$

Note: If we tried to solve $y^2 = x$ for y, we have $y = \pm\sqrt{x}$. The $\pm$ out front make it so we can't find the derivative of $y = \pm\sqrt{x}$ as written.

$$\boxed{\frac{d}{dx}(f(y)) = f'(y)\,\frac{dy}{dx}}$$

* In general, for implicit differentiation, take the derivative of any function of y as it appears then multiply by $\frac{dy}{dx}$ or y' (whichever you prefer).

Ex. 2 Find $\frac{dy}{dx}$ for $x^2 + 2xy + 2xy^2 = x^2y + y^3$.

HW#2
HW#3

① Take derivative of both sides with respect to x.

$$\frac{d}{dx}(x^2 + 2xy + 2xy^2) = \frac{d}{dx}(x^2y + y^3)$$

$$\underbrace{\frac{d}{dx}(x^2)}_{=2x} + 2\underbrace{\frac{d}{dx}(xy)}_{\text{Product}} + 2\underbrace{\frac{d}{dx}(xy^2)}_{\text{Product}} = \underbrace{\frac{d}{dx}(x^2y)}_{\text{Product}} + \underbrace{\frac{d}{dx}(y^3)}_{3y^2\cdot\frac{dy}{dx}}$$

"derivative of what you see"

- $$\begin{aligned}\frac{d}{dx}(xy) &= \frac{d}{dx}(x)\cdot y + x\cdot\frac{d}{dx}(y)\\ &= (1)\cdot y + x\frac{dy}{dx}\\ &= y + x\frac{dy}{dx}\end{aligned}$$

- $$\begin{aligned}\frac{d}{dx}(xy^2) &= \frac{d}{dx}(x)\cdot y^2 + x\cdot\frac{d}{dx}(y^2)\\ &= (1)\cdot y^2 + x\left(2y\frac{dy}{dx}\right)\\ &= y^2 + 2xy\frac{dy}{dx}\end{aligned}$$

- $$\begin{aligned}\frac{d}{dx}(x^2y) &= \frac{d}{dx}(x^2)\cdot y + x^2\frac{d}{dx}(y)\\ &= 2xy + x^2\frac{dy}{dx}\end{aligned}$$

$$2x + 2\left(y + x\frac{dy}{dx}\right) + 2\left(y^2 + 2xy\frac{dy}{dx}\right) = \left(2xy + x^2\frac{dy}{dx}\right) + 3y^2\frac{dy}{dx}$$

$$2x + 2y + 2x\frac{dy}{dx} + 2y^2 + 4xy\frac{dy}{dx} = 2xy + x^2\frac{dy}{dx} + 3y^2\frac{dy}{dx}$$

② Move all terms with $\frac{dy}{dx}$ to one side and the rest to the other side.

$$2x\frac{dy}{dx} + 4xy\frac{dy}{dx} - x^2\frac{dy}{dx} - 3y^2\frac{dy}{dx} = 2xy - 2x - 2y - 2y^2$$

③ Factor $\frac{dy}{dx}$ from all terms on (in this case) the left side.

$$(2x + 4xy - x^2 - 3y^2)\frac{dy}{dx} = 2xy - 2x - 2y - 2y^2$$

④ Divide both sides by everything next to $\frac{dy}{dx}$.

$$\frac{\cancel{(2x + 4xy - x^2 - 3y^2)}\frac{dy}{dx}}{\cancel{(2x + 4xy - x^2 - 3y^2)}} = \frac{2xy - 2x - 2y - 2y^2}{(2x + 4xy - x^2 - 3y^2)}$$

$$\boxed{\frac{dy}{dx} = \frac{2xy - 2x - 2y - 2y^2}{2x + 4xy - x^2 - 3y^2}}$$

* In LON-CAPA, $4xy = \boxed{4 * x * y}$

This asterisk is <u>required</u>.

Ex. 3 Find $\frac{dy}{dx}$ for $\sqrt{y} + x^2 = 14 + \sqrt{x} + y^2$ at $(4,1)$.
(Find the slope of the tangent line to $\sqrt{y} + x^2 = 14 + \sqrt{x} + y^2$ at $(4,1)$.) same question 2 ways.

HW#1
HW#4
HW#6

① Take derivative of both sides with respect to x to find $\frac{dy}{dx}$.

$$\frac{d}{dx}(y^{1/2} + x^2) = \frac{d}{dx}(14 + x^{1/2} + y^2)$$
$$\frac{d}{dx}(y^{1/2}) + \frac{d}{dx}(x^2) = \frac{d}{dx}(14) + \frac{d}{dx}(x^{1/2}) + \frac{d}{dx}(y^2)$$
$$\frac{1}{2}y^{-1/2}\frac{dy}{dx} + 2x = 0 + \frac{1}{2}x^{-1/2} + 2y\frac{dy}{dx}$$
$$\frac{1}{2\sqrt{y}}\frac{dy}{dx} + 2x = \frac{1}{2\sqrt{x}} + 2y\frac{dy}{dx}$$ (Could also evaluate at this step.)

② Move all $\frac{dy}{dx}$ terms to one side and the rest to the other.

$$\frac{1}{2\sqrt{y}}\frac{dy}{dx} - 2y\frac{dy}{dx} = \frac{1}{2\sqrt{x}} - 2x$$

③ Factor out $\frac{dy}{dx}$.

$$\left(\frac{1}{2\sqrt{y}} - 2y\right)\frac{dy}{dx} = \frac{1}{2\sqrt{x}} - 2x$$

④ Divide both sides by everything next to $\frac{dy}{dx}$.

$$\frac{dy}{dx} = \frac{\frac{1}{2\sqrt{x}} - 2x}{\frac{1}{2\sqrt{y}} - 2y}$$

⑤ Evaluate at $(4,1) \Rightarrow x = 4$ and $y = 1$.

$$\left.\frac{dy}{dx}\right|_{(4,1)} = \frac{\frac{1}{2\sqrt{4}} - 2(4)}{\frac{1}{2\sqrt{1}} - 2(1)} = \frac{\frac{1}{4} - 8}{\frac{1}{2} - 2} = \frac{\frac{1}{4} - \frac{32}{4}}{\frac{1}{2} - \frac{4}{2}} = \frac{-\frac{31}{4}}{-\frac{3}{2}} = \left(-\frac{31}{4_2}\right)\left(-\frac{2}{3}\right)$$
$$= \boxed{\frac{31}{6}}$$

Ex. 4 Find the slope of the tangent line to $\frac{3}{x} + \frac{1}{4y} = 10$ at $(1, \frac{1}{28})$.

HW#5

① Take the derivative of both sides with respect to x to find $\frac{dy}{dx}$.

$$\frac{d}{dx}\left(\frac{3}{x} + \frac{1}{4y}\right) = \frac{d}{dx}(10)$$ ← Note: By writing $\frac{1}{4y}$ as $\frac{1}{4}\cdot\frac{1}{y}$, we don't have to do a chain rule for $\frac{1}{4y}$: Out $= \frac{1}{y}$ In $= 4y$

$$\frac{d}{dx}\left(\frac{3}{x}\right) + \frac{d}{dx}\left(\frac{1}{4}\cdot\frac{1}{y}\right) = 0$$
$$3\frac{d}{dx}(x^{-1}) + \frac{1}{4}\frac{d}{dx}(y^{-1}) = 0$$ ← By writing $\frac{1}{y} = y^{-1}$, we avoid a quotient rule.
$$3(-x^{-1-1}) + \frac{1}{4}(-y^{-1-1})\frac{dy}{dx} = 0$$
$$-3x^{-2} - \frac{1}{4}y^{-2}\frac{dy}{dx} = 0$$
$$-\frac{3}{x^2} - \frac{1}{4y^2}\frac{dy}{dx} = 0$$

* $\frac{dy}{dx}$ should never end up in a denominator. *

② Evaluate at $(1, \frac{1}{28}) \Rightarrow \begin{matrix} x=1 \\ y=\frac{1}{28} \end{matrix}$. Solve for $\frac{dy}{dx}$.

$$-\frac{3}{(1)^2} - \frac{1}{4(\frac{1}{28})^2}\cdot\frac{dy}{dx} = 0$$

$$-3 - \frac{1}{4(\frac{1}{784})}\cdot\frac{dy}{dx} = 0$$

$$\cancel{\frac{4}{784}}\left[-\frac{1}{\left(\frac{4}{784}\right)}\cdot\frac{dy}{dx} = 3\right]\frac{4}{784}$$

$$-1\cdot\frac{dy}{dx} = 3\cdot\frac{4}{784}$$

$$\frac{dy}{dx} = -\frac{12}{784} = \boxed{-\frac{3}{196}}$$

Ex.5 Find $\frac{dy}{dx}$ for $2\tan(x)\cos(y) = 10$.

HW#7 ① Take the derivative of both sides with respect to x.

$$\frac{d}{dx}\big(\underbrace{2\tan(x)\cdot\cos(y)}_{\text{product}}\big) = \frac{d}{dx}(10)$$

$$\underbrace{\frac{d}{dx}(2\tan(x))}_{=2\sec^2(x)}\cdot\cos(y) + 2\tan(x)\cdot\underbrace{\frac{d}{dx}(\cos(y))}_{=-\sin(y)\cdot\frac{dy}{dx}} = 0$$

$$2\sec^2(x)\cos(y) + 2\tan(x)\left[-\sin(y)\frac{dy}{dx}\right] = 0$$

$$2\sec^2(x)\cos(y) - 2\tan(x)\sin(y)\frac{dy}{dx} = 0$$

② Put $\frac{dy}{dx}$ and "non" $\frac{dy}{dx}$ terms on opposite sides.

$$2\sec^2(x)\cos(y) = 2\tan(x)\sin(y)\frac{dy}{dx}$$

③ Factor out $\frac{dy}{dx}$. ✓Already done

④ Divide by coefficient on $\frac{dy}{dx}$.

$$\frac{\cancel{2}\sec^2(x)\cos(y)}{\cancel{2}\tan(x)\sin(y)} = \frac{dy}{dx}$$

$$\boxed{\frac{\sec^2(x)\cos(y)}{\tan(x)\sin(y)} = \frac{dy}{dx}}$$

* Note: When trig functions have the same arguments (stuff in parentheses), we can try to rewrite them:

- $\frac{\cos(y)}{\sin(y)} = \cot(y)$
- $\frac{\sec^2(x)}{\tan(x)} = \frac{\left(\frac{1}{\cos^2(x)}\right)}{\left(\frac{\sin(x)}{\cos(x)}\right)} = \frac{1}{\cos^2(x)}\cdot\frac{\cos(x)}{\sin(x)} = \frac{1}{\cos(x)}\cdot\frac{1}{\sin(x)}$
  $= \sec(x)\csc(x)$

So we can also write $\boxed{\frac{dy}{dx} = \sec(x)\csc(x)\cot(y)}$ in Ex.5.

* $\frac{dy}{dx}$ should not appear inside trig functions *

Ex. 6 Find $\frac{dy}{dx}$ for $\tan(2x+3y) = 2x^2y$.

HW#8

① Take derivative of both sides with respect to x.

$$\underbrace{\frac{d}{dx}(\tan(2x+3y))}_{\text{Chain}} = \underbrace{\frac{d}{dx}(x^2y)}_{\text{Product}}$$

• Chain: Out $= \tan(x)$ In $= 2x+3y$

Out' $= \sec^2(x)$

$$\begin{aligned} \text{In}' &= \frac{d}{dx}(2x+3y) \\ &= \frac{d}{dx}(2x) + \frac{d}{dx}(3y) \\ &= 2 + 3\frac{dy}{dx} \end{aligned}$$

$$\begin{aligned} \frac{d}{dx}(\tan(2x+3y)) &= \text{Out}'(\text{In}) \cdot \text{In}' \\ &= \sec^2(\text{In}) \cdot \left(2+3\frac{dy}{dx}\right) \\ &= \sec^2(2x+3y) \cdot \left(2+3\frac{dy}{dx}\right) \end{aligned}$$

• Product: $\frac{d}{dx}(x^2y) = \frac{d}{dx}(x^2) \cdot y + x^2 \cdot \frac{d}{dx}(y)$

$$= 2xy + x^2\frac{dy}{dx}$$

$$\frac{d}{dx}(\tan(2x+3y)) = \frac{d}{dx}(x^2y)$$

$$\underbrace{\sec^2(2x+3y)\left(2+3\frac{dy}{dx}\right)}_{\text{Need to distribute before we can solve for } \frac{dy}{dx}.} = 2xy + x^2\frac{dy}{dx}$$

$$2\sec^2(2x+3y) + 3\frac{dy}{dx}\sec^2(2x+3y) = 2xy + x^2\frac{dy}{dx}$$

② Put all $\frac{dy}{dx}$ terms on their own side.

$$3\frac{dy}{dx}\sec^2(2x+3y) - x^2\frac{dy}{dx} = 2xy - 2\sec^2(2x+3y)$$

③ Factor out $\frac{dy}{dx}$.

$$\left(3\sec^2(2x+3y) - x^2\right)\frac{dy}{dx} = 2xy - 2\sec^2(2x+3y)$$

④ Divide by coefficient on $\frac{dy}{dx}$.

$$\boxed{\frac{dy}{dx} = \frac{2xy - 2\sec^2(2x+3y)}{3\sec^2(2x+3y) - x^2}}$$

Ex.7 Find $\frac{dy}{dx}$ for $3\sec\left(\frac{x^2}{y}\right) = x$.

HW 9

① Take derivative of both sides with respect to x.

$$\underbrace{\frac{d}{dx}\left(3\sec\left(\frac{x^2}{y}\right)\right)}_{\text{Chain}} = \underbrace{\frac{d}{dx}(x)}_{=1}$$

- Chain: Out $= 3\sec(x)$
  Out$' = 3\sec(x)\tan(x)$

  In $= \underbrace{\frac{x^2}{y}}$

  Quotient Rule:

  Top $= x^2$ Bottom $= y$

  Top$' = 2x$ Bottom$' = \frac{dy}{dx}$

$$\text{In}' = \frac{\text{Top}' \cdot \text{Bottom} - \text{Top} \cdot \text{Bottom}'}{\text{Bottom}^2} = \frac{2xy - x^2\frac{dy}{dx}}{y^2}$$

$$\frac{d}{dx}\left(3\sec\left(\frac{x^2}{y}\right)\right) = \text{Out}'(\text{In}) \cdot \text{In}'$$

$$= 3\sec(\text{In})\tan(\text{In}) \cdot \left(\frac{2xy - x^2\frac{dy}{dx}}{y^2}\right)$$

$$= 3\sec\left(\frac{x^2}{y}\right)\tan\left(\frac{x^2}{y}\right)\frac{\left(2xy - x^2\frac{dy}{dx}\right)}{y^2}$$

$$= \frac{3\left(2xy - x^2\frac{dy}{dx}\right)\sec\left(\frac{x^2}{y}\right)\tan\left(\frac{x^2}{y}\right)}{y^2}$$

$$\frac{d}{dx}\left(3\sec\left(\frac{x^2}{y}\right)\right) = 1$$

$$\cancel{y^2}\left[\frac{3\left(2xy - x^2\frac{dy}{dx}\right)\sec\left(\frac{x^2}{y}\right)\tan\left(\frac{x^2}{y}\right)}{\cancel{y^2}} = 1\right] y^2$$

$$3\underbrace{\left(2xy - x^2\frac{dy}{dx}\right)}\sec\left(\frac{x^2}{y}\right)\tan\left(\frac{x^2}{y}\right) = y^2$$

Distribute before we can solve for $\frac{dy}{dx}$.

$$6xy\sec\left(\frac{x^2}{y}\right)\tan\left(\frac{x^2}{y}\right) - 3x^2\frac{dy}{dx}\sec\left(\frac{x^2}{y}\right)\tan\left(\frac{x^2}{y}\right) = y^2$$

② Bring all $\frac{dy}{dx}$ terms to one side

$$-3x^2\frac{dy}{dx}\sec\left(\frac{x^2}{y}\right)\tan\left(\frac{x^2}{y}\right) = y^2 - 6xy\sec\left(\frac{x^2}{y}\right)\tan\left(\frac{x^2}{y}\right)$$

③ Factor out $\frac{dy}{dx}$. ✓

④ Divide by coefficient on $\frac{dy}{dx}$.

$$\boxed{\frac{dy}{dx} = \frac{y^2 - 6xy\sec\left(\frac{x^2}{y}\right)\tan\left(\frac{x^2}{y}\right)}{-3x^2\sec(x^2/y)\tan(x^2/y)}}$$

Ex.8 Find $\frac{dy}{dx}$ for $e^{x^2y} = 2y$.

HW#10

① Take derivative of both sides with respect to x.

$$\underbrace{\frac{d}{dx}(e^{x^2y})}_{\text{Chain}} = \underbrace{\frac{d}{dx}(2y)}_{=2\frac{dy}{dx}}$$

• Chain: Out $= e^x$, Out$' = e^x$ In $= \underbrace{x^2y}_{\text{Product}}$

$$In' = \frac{d}{dx}(x^2)\cdot y + x^2\frac{d}{dx}(y) = 2xy + x^2\frac{dy}{dx}$$

$$\frac{d}{dx}(e^{x^2y}) = Out'(In)\cdot In' = e^{In}\left(2xy + x^2\frac{dy}{dx}\right) = e^{x^2y}\left(2xy + x^2\frac{dy}{dx}\right)$$

$$\frac{d}{dx}(e^{x^2y}) = 2\frac{dy}{dx}$$

$$e^{x^2y}\left(2xy + x^2\underbrace{\frac{dy}{dx}}\right) = 2\frac{dy}{dx}$$

$\frac{dy}{dx}$ is trapped in the parentheses, so we have to distribute.

$$2xye^{x^2y} + x^2\frac{dy}{dx}e^{x^2y} = 2\frac{dy}{dx}$$

② Put all $\frac{dy}{dx}$ terms on one side.

$$2xye^{x^2y} = 2\frac{dy}{dx} - x^2\frac{dy}{dx}e^{x^2y}$$

③ Factor out $\frac{dy}{dx}$.

$$2xye^{x^2y} = (2 - x^2e^{x^2y})\frac{dy}{dx}$$

④ Divide by coefficient on $\frac{dy}{dx}$

$$\boxed{\frac{2xye^{x^2y}}{2 - x^2e^{x^2y}} = \frac{dy}{dx}}$$

* $\frac{dy}{dx}$ should never be in the exponent. *