Lesson 15: Related Rates

Ex. 1 Assume x and y are both differentiable functions of t and $x^2 + y^2 = 4$. Find $\frac{dx}{dt}$ if $\frac{dy}{dt} = 1,\ x = 1$, and y is positive.

$$\frac{d}{dt}(x^2 + y^2 = 4)$$

$$\underbrace{\frac{d}{dt}(x^2)}_{\text{chain}} + \underbrace{\frac{d}{dt}(y^2)}_{\text{chain}} = \frac{d}{dt}(4)$$

since both are functions of t

$Out = x^2 \quad In = x$

$Out' = 2x \quad In' = \frac{dx}{dt}$

$$\frac{d}{dt}(x^2) = 2x\frac{dx}{dt}$$

$Out = y^2 \quad In = y$

$Out' = 2y \quad In' = \frac{dy}{dt}$

$$\frac{d}{dt}(y^2) = 2y\frac{dy}{dt}$$

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$

$$2(1)\frac{dx}{dt} + 2y(1) = 0$$

what we are looking for

can use original $x^2+y^2=4$ to find when $x=1$ and y positive

$1^2 + y^2 = 4$

$y^2 = 3$

$y = \pm\sqrt{3}$

$y = \sqrt{3}$

$$2\frac{dx}{dt} + 2\sqrt{3} = 0$$

$$2\frac{dx}{dt} = -2\sqrt{3}$$

$$\boxed{\frac{dx}{dt} = -\sqrt{3}}$$

Ex. 2 If x and y are both functions of t and $x + y^2 = 4\,e^x$, find $\frac{dy}{dt}$ when $\frac{dx}{dt} = 2,\ x = 0$, and $y = -2$.

$$\frac{d}{dt}(x + y^2 = 4e^x)$$

$$\frac{d}{dt}(x) + \frac{d}{dt}(y^2) = \frac{d}{dt}(4e^x)$$

$$(x)'\frac{dx}{dt} + (y^2)'\frac{dy}{dt} = (4e^x)'\frac{dx}{dt}$$

$$1\cdot\frac{dx}{dt} + 2y\cdot\frac{dy}{dt} = 4e^x\frac{dx}{dt}$$

$$\frac{dx}{dt} + 2y\frac{dy}{dt} = 4e^x\frac{dx}{dt}$$

$$(2) + 2(-2)\left(\frac{dy}{dt}\right) = 4e^0(2)$$

$$2 - 4\frac{dy}{dt} = 4(1)(2) = 8$$

$$-4\frac{dy}{dt} = 6$$

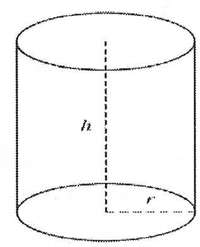
$$\boxed{\frac{dy}{dt} = -\frac{3}{2}}$$

Word Problems

Suggested Steps:

1. Determine the necessary formula. (Volume, area, perimeter, surface area, etc.)
2. Identify which quantities change with respect to ***time*** and which are ***constant***.
3. If a quantity is ***constant***, plug the value into the formula.
4. Determine which rates of change are given.
   If a quantity changes with respect to time and its rate of change is **not** given, we need to relate it to another variable.
   Use this relation to write the formula in terms of variables that change with respect to time and whose rates of change are given OR are what we want to solve for.
   (For instance, if sand is being poured and forming a conical pile, the height, radius and volume of the pile are changing. This means we need the rates of change for all three quantities. If we're not given one or two of them, we have to find a way to relate them.)
5. **Take the derivative of both sides of the formula with respect to *time*. (Use implicit differentiation)**
6. Plug in the given values and solve for the desired quantity.

<u>**Ex. 3**</u> Kel is filling a cylindrical glass of diameter 10 inches with orange soda (because Kel loves orange soda). How fast does the soda level drop when he's drinking it (through a straw) at a rate of $3\ \text{in}^3/\text{sec}$? (Recall: Volume of a cylinder is $V = \pi r^2 h$, where r is the radius and h is the height.)



Hint: Does r change with respect to time?

① $V = \pi r^2 h$

② V - changes with t
r - constant = 5in
h - changes with t

③ $V = \pi (5)^2 h = 25\pi h$

④ Need $\frac{dV}{dt}$ and $\frac{dh}{dt}$

$\frac{dV}{dt}$ ↓ given = $3\text{in}^3/\text{sec}$; $\frac{dh}{dt}$ ↓ looking for

No relation needed.

⑤ $\frac{d}{dt}(V = 25\pi h)$

$$\frac{dV}{dt} = 25\pi \frac{dh}{dt}$$

⑥ $$3 = 25\pi \frac{dh}{dt}$$

$$\boxed{\frac{3}{25\pi} = \frac{dh}{dt}}$$

Ex. 4 Rubber bands are being added to a large rubber band ball so that the radius of the ball is increasing at a rate of 1.2 cm/min.

(a) How fast is the volume increasing when $r = 10$ cm? (Recall: Volume of a sphere is $V = \frac{4}{3}\pi r^3$, where r is the radius of the sphere.)

① $V = \frac{4}{3}\pi r^3$

② V - changes with t
r - changes with t

③ No constant quantity

④ $\frac{dV}{dt}$ - looking for
$\frac{dr}{dt}$ - given $= 1.2 = \frac{12}{10}$
$\Rightarrow$ No relation needed

⑤ $\frac{d}{dt}\left(V = \frac{4}{3}\pi r^3\right)$

$$\frac{dV}{dt} = \frac{4}{3}\pi (r^3)' \frac{dr}{dt}$$

$$\frac{dV}{dt} = \frac{4}{\cancel{3}}\pi (\cancel{3}r^2) \frac{dr}{dt}$$

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$$

⑥ $\frac{dV}{dt} = 4\pi (10)^2 \left(\frac{12}{\cancel{10}}\right)$

$$\frac{dV}{dt} = 4\pi (10)(12)$$

$$\boxed{\frac{dV}{dt} = 480\pi}$$

(b) How fast is the surface area increasing when $r = 10$ cm? (Recall: Surface area of a sphere is $S = 4\pi r^2$, where r is the radius of the sphere.)

① $S = 4\pi r^2$

② S - changes with t
r - changes with t

③ No constant quantity.

④ $\frac{dS}{dt}$ - looking for
$\frac{dr}{dt}$ - given $= \frac{12}{10}$

⑤ $\frac{d}{dt}\left(S = 4\pi r^2\right)$

$$\frac{dS}{dt} = 4\pi (r^2)' \frac{dr}{dt}$$

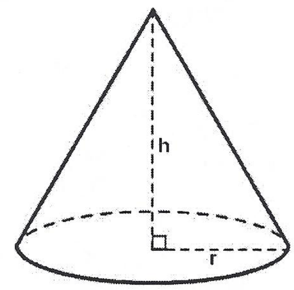
$$\frac{dS}{dt} = 4\pi (2r) \frac{dr}{dt}$$

$$\frac{dS}{dt} = 8\pi r \frac{dr}{dt}$$

⑥ $\frac{dS}{dt} = 8\pi (\cancel{10}) \left(\frac{12}{\cancel{10}}\right)$

$$\boxed{\frac{dS}{dt} = 96\pi}$$

Ex. 5 John Sheppard is pouring hair gel into his hand at $1 \text{ cm}^3/\text{sec}$, forming a conical glob whose base diameter is always three times its altitude. How fast is the altitude of the pile increasing when the pile is 1 cm high? (Recall: Volume of a cone is $V = \frac{1}{3}\pi r^2 h$, where r is the radius of the base and h is the height.)



① $V = \frac{1}{3}\pi r^2 h$

② V - changes with t
r - changes with t
h - changes with t

③ No constant quantities.

④ $\frac{dV}{dt}$ - given $= 1 \text{ cm}^3/\text{s}$
$\frac{dr}{dt}$ - not given or sought
$\frac{dh}{dt}$ - sought

$\Rightarrow$ Need to relate r to V or h.
Diameter is always three times the altitude.

$$(2r) = 3 \cdot (h)$$

$$2r = 3h$$

$$r = \frac{3}{2}h$$

Plug into equation:

$$V = \frac{1}{3}\pi r^2 h$$

$$V = \frac{1}{3}\pi \left(\frac{3}{2}h\right)^2 h$$

$$V = \frac{1}{3}\pi \left(\frac{3}{2}\right)^2 h^2 h$$

$$V = \frac{1}{3}\pi \frac{9}{4} h^3$$

$$V = \frac{3}{4}\pi h^3$$

⑤ $\frac{d}{dt}\left(V = \frac{3}{4}\pi h^3\right)$

$$\frac{dV}{dt} = \frac{3}{4}\pi (h^3)' \frac{dh}{dt}$$

$$\frac{dV}{dt} = \frac{3}{4}\pi (3h^2) \frac{dh}{dt}$$

$$\frac{dV}{dt} = \frac{9}{4}\pi h^2 \frac{dh}{dt}$$

⑥ $1 = \frac{9}{4}\pi (1)^2 \frac{dh}{dt}$

$$\frac{4}{9}\left[1 = \frac{9}{4}\pi \frac{dh}{dt}\right]$$

$$\frac{4}{9} = \pi \frac{dh}{dt}$$

$$\boxed{\frac{4}{9\pi} = \frac{dh}{dt}}$$