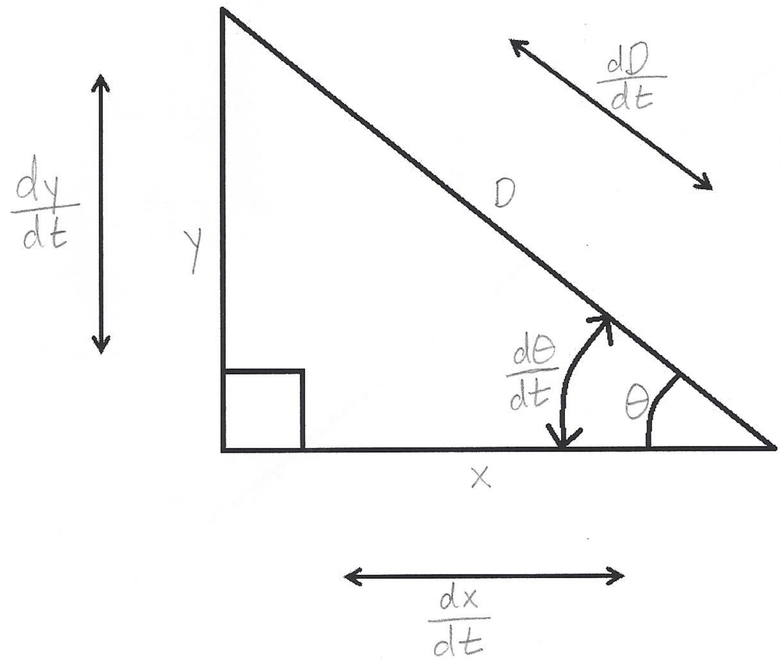
Lesson 16: More Related Rates



For this section, we're considering right triangles, so we'll mostly be using the Pythagorean Theorem. For the triangle above, we have $x^2 + y^2 = D^2$.

We also need to remember some of our trig functions:

$\sin(\theta) = \frac{y}{D}$ $\cos(\theta) = \frac{x}{D}$ $\tan(\theta) = \frac{y}{x}$

Suggested Steps:

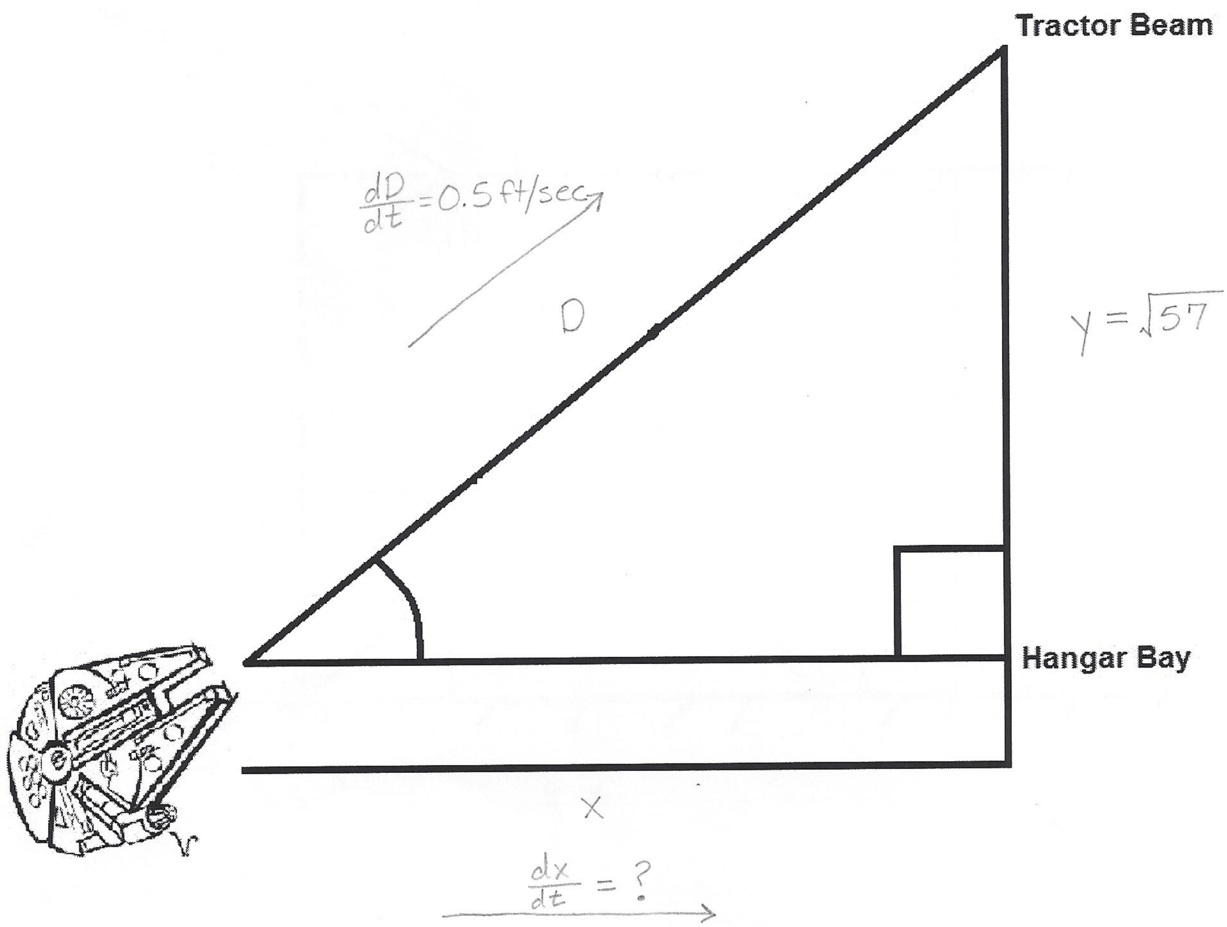
1. Read the problem in its entirety.
2. Draw the figure, and add necessary arrows to indicate motion.

   Label the figure with any given or desired variables (x, y, D, θ) and arrows with rates of change $\left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dD}{dt}, \frac{d\theta}{dt}\right)$.
3. Pick the appropriate relation between the quantities.

   *For **angles**, pick a relation where one value is **constant** and the **rate** of the other is given.*
4. If a quantity is *constant*, plug it into the equation.
5. Take the derivative of both sides of the equation with respect to time.
6. Plug in the given information, and solve for the desired quantity.

   (If any information is missing, use the **original relation** to find it.)

Ex. 1 The Millennium Falcon is being pulled into a hangar bay of the Death Star by a tractor beam located $\sqrt{57}$ ft higher than the hangar bay floor. (This vertical distance remains constant as the Falcon moves towards the hangar bay.) The beam pulls at a rate of 0.5 ft/sec. How fast is the horizontal distance between the Falcon and the hangar bay decreasing when the beam is 11 ft "long"?



$$x^2 + y^2 = D^2$$
$$x^2 + (\sqrt{57})^2 = D^2$$
$$x^2 + 57 = D^2$$
$$\frac{d}{dt}(x^2 + 57) = \frac{d}{dt}(D^2)$$
$$\frac{d}{dt}(x^2) + \frac{d}{dt}(57) = \frac{d}{dt}(D^2)$$
$$2x\frac{dx}{dt} + 0 = 2D\frac{dD}{dt}$$
$$2x\frac{dx}{dt} = 2D\frac{dD}{dt}$$
$$x\frac{dx}{dt} = D\frac{dD}{dt}$$

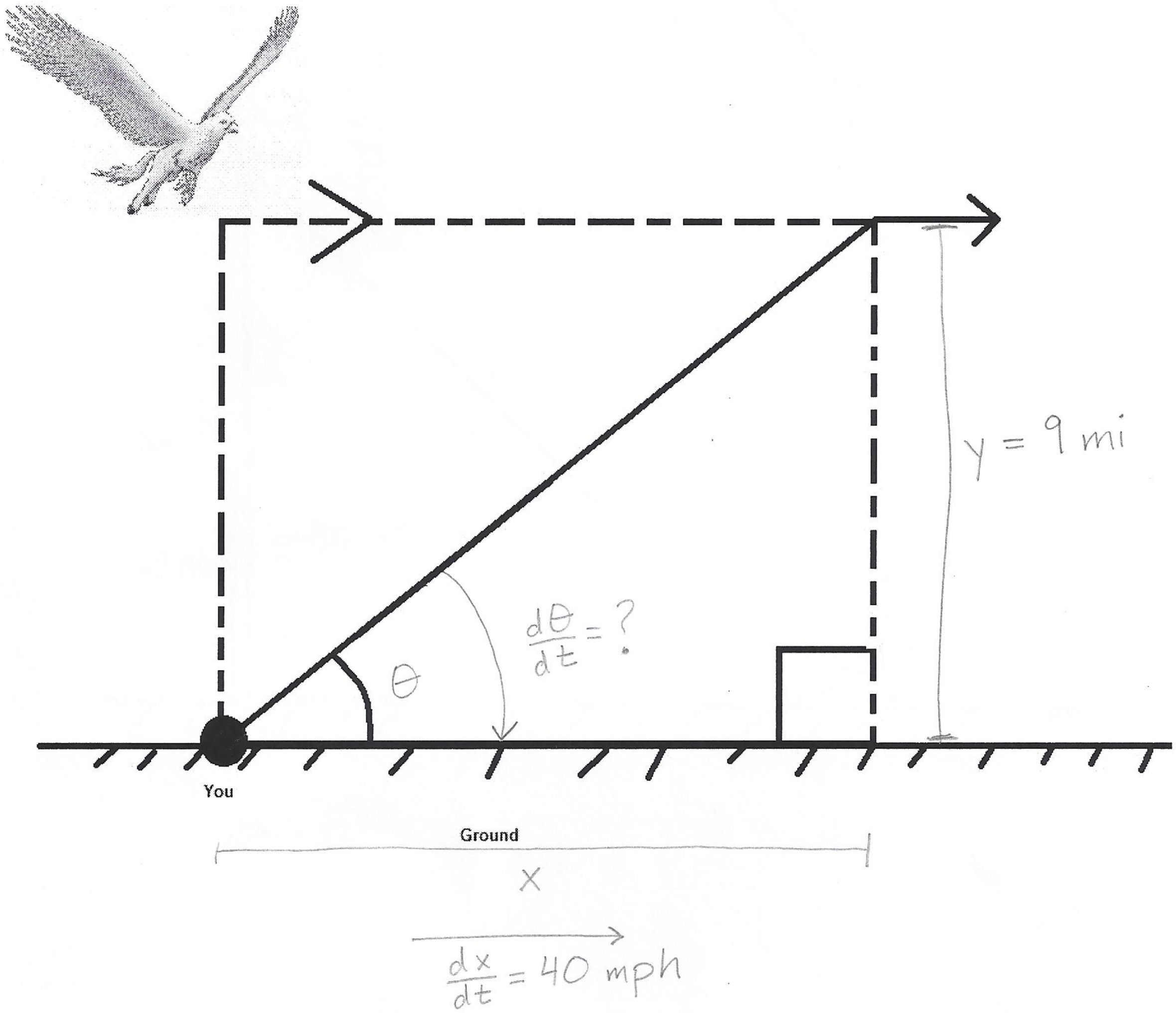
need wanted 11 0.5

To find x, use $x^2 + y^2 = D^2$ when $D = 11$:

$$x^2 + (\sqrt{57})^2 = 11^2$$
$$x^2 + 57 = 121$$
$$x^2 = 64$$
$$x = 8$$

$$8\frac{dx}{dt} = 11\left(\frac{1}{2}\right)$$
$$\frac{dx}{dt} = 11\left(\frac{1}{2}\right)\left(\frac{1}{8}\right) = \boxed{\frac{11}{16}\text{ ft/sec}}$$

Ex. 2 A hippogriff is flying directly away from you at 40 mph at an altitude of 9 miles. Find the rate at which the angle of elevation is decreasing when the angle is $\frac{\pi}{6}$.



y is constant and $\frac{dx}{dt}$ is given, so we should choose $\tan\theta$.

$$\tan(\theta) = \frac{y}{x}$$

$$\tan(\theta) = \frac{9}{x} = 9x^{-1}$$

$$\frac{d}{dt}(\tan(\theta)) = \frac{d}{dt}(9x^{-1})$$

$$\sec^2(\theta)\frac{d\theta}{dt} = 9(-x^{-2})\frac{dx}{dt}$$

$$\underset{\theta = \frac{\pi}{6}}{(\sec(\theta))^2}\ \underset{\text{wanted}}{\frac{d\theta}{dt}} = -\underset{\text{need } x}{\frac{9}{x^2}}\ \underset{40}{\frac{dx}{dt}}$$

Use $\tan(\theta) = \frac{y}{x}$ to find x:

$$\tan\left(\frac{\pi}{6}\right) = \frac{9}{x}$$

$$\frac{\sin\left(\frac{\pi}{6}\right)}{\cos\left(\frac{\pi}{6}\right)} = \frac{9}{x}$$

$$\frac{1/2}{\sqrt{3}/2} = \frac{9}{x}$$

$$\frac{1}{\sqrt{3}} = \frac{9}{x}$$

$$x = 9\sqrt{3}$$

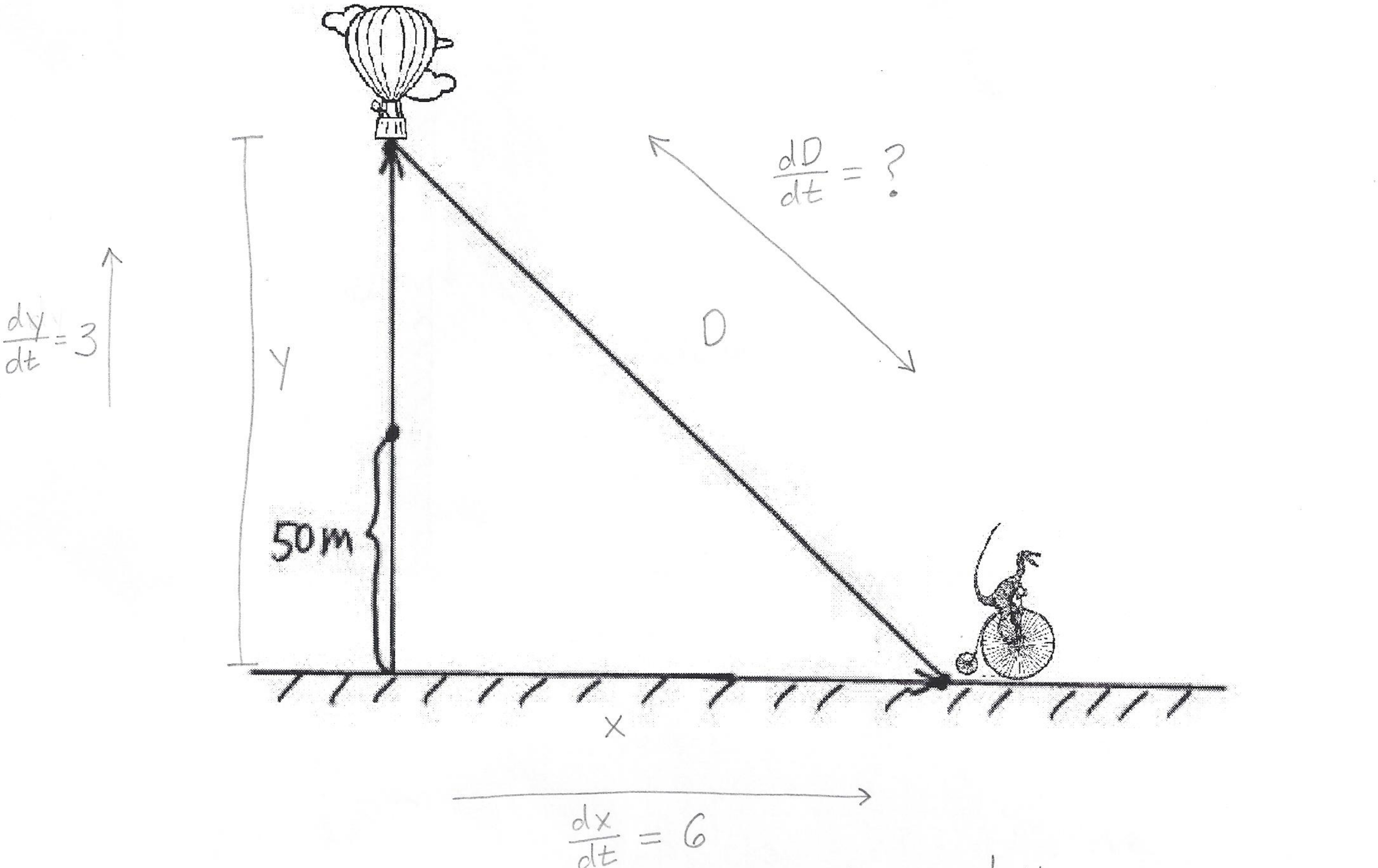
$$\frac{d\theta}{dt} = -\frac{9}{(9\sqrt{3})^2}(40)\left(\cos\left(\frac{\pi}{6}\right)\right)^2$$

$$\frac{d\theta}{dt} = -\frac{\cancel{9}}{\cancel{81}_{9}\cdot 3}(40)\left(\frac{\sqrt{3}}{2}\right)^2 = -\frac{1}{\cancel{27}_{9}}\cdot \overset{10}{\cancel{40}}\cdot\frac{\cancel{3}}{\cancel{4}} = -\frac{10}{9}$$

All answers should be positive.

$$\boxed{\frac{10}{9} \text{ rad/sec}}$$

Ex. 3 A hot air balloon is at a height of 50 meters and is rising at the constant rate of 3 m/sec. A velociraptor on a bicycle passes beneath it traveling in a straight line at the constant speed of 6 m/sec. How fast is the distance between the velociraptor and the hot air balloon increasing 10 seconds later?



$x^2 + y^2 = D^2$

$\frac{d}{dt}(x^2 + y^2) = \frac{d}{dt}(D^2)$

$\frac{d}{dt}(x^2) + \frac{d}{dt}(y^2) = \frac{d}{dt}(D^2)$

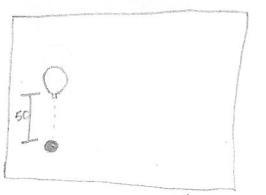
$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 2D\frac{dD}{dt}$

$\underbrace{x}_{\text{Need}}\underbrace{\frac{dx}{dt}}_{6} + \underbrace{y}_{\text{Need}}\underbrace{\frac{dy}{dt}}_{3} = \underbrace{D}_{\text{Need}}\underbrace{\frac{dD}{dt}}_{\text{Wanted}}$

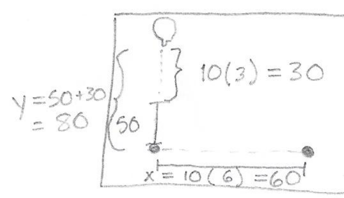
To find x and y,

Recall: distance = rate × time

@ t = 0



@ t = 10



To find D, use $x^2 + y^2 = D^2$:

$x^2 + y^2 = D^2$

$(60)^2 + (80)^2 = D^2$

$10000 = D^2$

$100 = D$

$(60)(6) + (80)(3) = 100\frac{dD}{dt}$

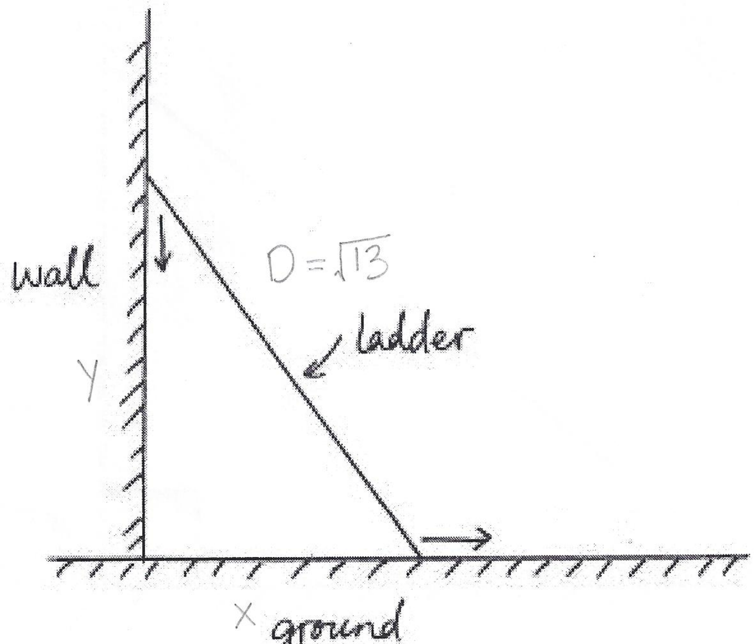
$360 + 240 = 100\frac{dD}{dt}$

$600 = 100\frac{dD}{dt}$

$\boxed{6 = \frac{dD}{dt}}$

Ex. 4 A ladder $\sqrt{13}$ meters long rests on horizontal ground and leans against a vertical wall. The foot of the ladder is pulled away from the wall at the rate of $0.3\ \mathrm{m/sec}$. How fast is the top sliding down the wall when the foot of the ladder is $3\ \mathrm{m}$ from the wall?

$\frac{dy}{dt} = ?$



$\frac{dx}{dt} = 0.3\ \mathrm{m/sec}$

$$x^2 + y^2 = D^2$$
$$x^2 + y^2 = (\sqrt{13})^2$$
$$x^2 + y^2 = 13$$
$$\frac{d}{dt}(x^2 + y^2) = \frac{d}{dt}(13)$$
$$\frac{d}{dt}(x^2) + \frac{d}{dt}(y^2) = 0$$
$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$
$$x\frac{dx}{dt} + y\frac{dy}{dt} = 0$$

3 0.3 Need Wanted

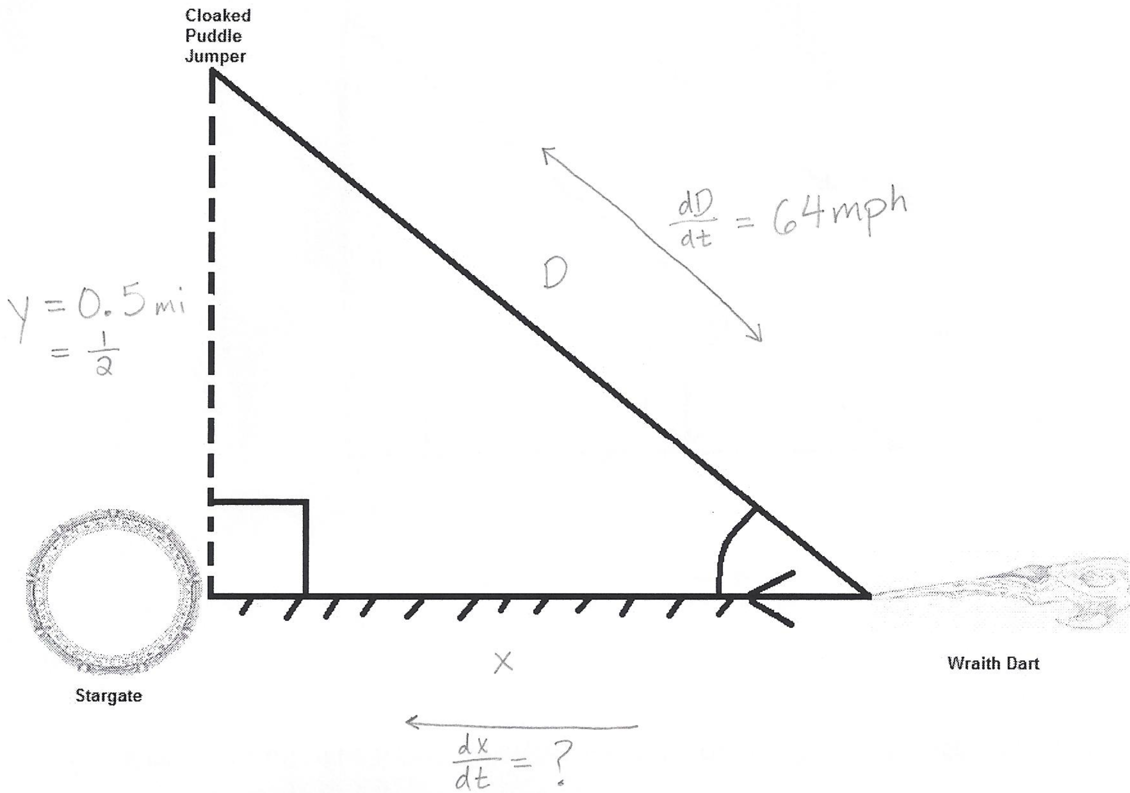
Use $x^2 + y^2 = D^2$ to find y:

$$x^2 + y^2 = D^2$$
$$(3)^2 + y^2 = (\sqrt{13})^2$$
$$9 + y^2 = 13$$
$$y^2 = 4$$
$$y = 2$$

$$(3)(0.3) + (2)\frac{dy}{dt} = 0$$
$$(3)\left(\frac{3}{10}\right) + 2\frac{dy}{dt} = 0$$
$$2\frac{dy}{dt} = -\frac{9}{10}$$
$$\frac{dy}{dt} = -\frac{9}{20}$$

$\boxed{\frac{9}{20}\ \mathrm{m/sec}}$

Ex. 5 A cloaked puddle jumper is hovering at a constant altitude of 0.5 miles above a Stargate. The jumper pilot uses radar to determine that an oncoming Wraith dart is at a distance of exactly 1 mile from the jumper, and that this distance is decreasing at 64 mph. Find the speed of the Wraith dart.



$$x^2 + y^2 = D^2$$
$$x^2 + \left(\tfrac{1}{2}\right)^2 = D^2$$
$$x^2 + \tfrac{1}{4} = D^2$$
$$\frac{d}{dt}\left(x^2 + \tfrac{1}{4}\right) = \frac{d}{dt}(D^2)$$
$$\frac{d}{dt}(x^2) + \frac{d}{dt}\left(\tfrac{1}{4}\right) = \frac{d}{dt}(D^2)$$
$$2x\frac{dx}{dt} + 0 = 2D\frac{dD}{dt}$$
$$2x\frac{dx}{dt} = 2D\frac{dD}{dt}$$
$$x\frac{dx}{dt} = D\frac{dD}{dt}$$

x: Need; $\frac{dx}{dt}$: wanted; D: 1; $\frac{dD}{dt}$: 64

To find x, use $x^2 + y^2 = D^2$:
$$x^2 + \left(\tfrac{1}{2}\right)^2 = (1)^2$$
$$x^2 + \tfrac{1}{4} = 1$$
$$x^2 = \tfrac{3}{4}$$
$$x = \sqrt{\tfrac{3}{4}} = \frac{\sqrt{3}}{2}$$

$$\frac{\sqrt{3}}{2}\frac{dx}{dt} = (1)(64)$$
$$\frac{dx}{dt} = 64 \cdot \frac{2}{\sqrt{3}} = \boxed{\frac{128}{\sqrt{3}} \text{ mph}}$$

Answers: 1. 11/16 ft/sec **2.** 10/9 rad/hour **3.** 6 m/sec **4.** 9/20 m/sec **5.** 128/$\sqrt{3}$ mph